MATH 410 Sections Final Exam 2

1. True or false: Determine if the following are true or false. If false provide a counterexample. You do not need to prove anything about the counterexamples, just provide them. [20 pts]
   (a) The product of two irrational numbers is irrational.
   (b) If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ then it is differentiable on (a, b).
   (c) The Taylor Polynomial for a polynomial equals that polynomial.
   (d) All polynomials are differentiable everywhere.
2. Suppose that $f : (a, b) \to \mathbb{R}$ is uniformly continuous. Show that f is bounded. [15 pts]
3. Prove using Mathematical Induction that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. [15 pts]
4. Using only the Archimedean Principle prove that $\left\{\frac{3}{\sqrt{n}} + \frac{1}{n^2} + 2\right\}$ converges. [15 pts]
5. Prove that $S \subseteq \mathbb{R}$ is dense iff $\forall x \in \mathbb{R}$ there is a sequence $\{x_n\}$ in S which converges to x. [15 pts]
6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $f(2) = 3$, $f'(2) = 0$ and $f''(x) \geq 3$ for all $x \in [0, 4]$. Find a lower bound on $f(4)$. [15 pts]
7. Define $f : [0, 2] \to \mathbb{R}$ by [10 pts]
$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}$$
   Using the limit definition of the derivative and the sequence definition of the limit prove that $f'(1)$ does not exist.
8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ and $f'(x_0) > 0$. Prove there is a neighborhood I of x_0 such that for $x \in I$ we have $x < x_0 \Rightarrow f(x) < f(x_0)$ and $x > x_0 \Rightarrow f(x) > f(x_0)$. [15 pts]
9. Suppose $f(4) = 3$ and $F(x) = \int_0^x (x - t)f(t^2)\ dt$. Find $F'(2)$. [10 pts]
10. Show that the Taylor expansion of $f(x) = \sin(4x)$ around any x_0 converges for all x. [10 pts]
11. Define $f_n : [3, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{1}{nx+1}$. Find the function $f : [3, \infty) \to \mathbb{R}$ to which $\{f_n\}$ converges pointwise and then show this convergence is uniform. [15 pts]
12. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \frac{1}{x+1}$. Using our proof of the Weierstrass Approximation Theorem find the minimum degree polynomial which approximates $f(x)$ uniformly within $\epsilon = 0.1$ and write this polynomial in Σ form. [15 pts]
13. Use Taylor Polynomials to prove that $\int_0^1 e^{(x^2)}\ dx \geq \frac{4}{3}$. [15 pts]
14. Show that $\sum_{k=1}^{\infty} \frac{3}{k^4 5^k}$ converges using the Weierstrass Convergence Criterion. [15 pts]