**MATH 410 Sections Final Exam 2 Hints**

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1. True or false: Determine if the following are true or false. If false provide a counterexample. You do not need to prove anything about the counterexamples, just provide them. [20 pts]

   (a) The product of two irrational numbers is irrational.
   **Hint:** False.

   (b) If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ then it is differentiable on $(a, b)$.
   **Hint:** False.

   (c) The Taylor Polynomial for a polynomial equals that polynomial.
   **Hint:** False.

   (d) All polynomials are differentiable everywhere.
   **Hint:** True.

2. Suppose that $f : (a, b) \to \mathbb{R}$ is uniformly continuous. Show that $f$ is bounded. [15 pts]
   **Hint:** Assume it's not bounded. Build a sequence and then invoke sequential compactness.

3. Prove using Mathematical Induction that $\sum\limits_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$. [15 pts]

   **Hint:** Straightforward.

4. Using only the Archimedian Principle prove that $\left\{ \frac{3}{\sqrt{n}} + \frac{1}{n^2} + 2 \right\}$ converges. [15 pts]
   **Hint:** Keep bounding until you can get $\frac{1}{n}$ less than something.

5. Prove that $S \subseteq \mathbb{R}$ is dense iff $\forall x \in \mathbb{R}$ there is a sequence $\{x_n\}$ in $S$ which converges to $x$. [15 pts]
   **Hint:** Make sure you know our definition of dense: $S$ is dense in $\mathbb{R}$ if for any interval $(a, b)$ with $a < b$ in the reals there is some $s \in S$ with $s \in (a, b)$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $f(2) = 3$, $f'(2) = 0$ and $f''(x) \geq 3$ for all $x \in [0, 4]$. Find a lower bound on $f(4)$. [15 pts]
   **Hint:** Use the formula we derived from the Cauchy MVT.

7. Define $f : [0, 2] \to \mathbb{R}$ by [10 pts]

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}$$

   Using the limit definition of the derivative and the sequence definition of the limit prove that $f'(1)$ does not exist.
   **Hint:** You're probably best off finding two sequences for which the corresponding limits are different.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ and $f'(x_0) > 0$. Prove there is a neighborhood $I$ of $x_0$ such that for $x \in I$ we have $x < x_0 \Rightarrow f(x) < f(x_0)$ and $x > x_0 \Rightarrow f(x) > f(x_0)$. [15 pts]
   **Hint:** By way of contradiction assume not and build a sequence. What can that sequence tell you about $f'(x_0)$?

9. Suppose $f(4) = 3$ and $F(x) = \int_0^x (x - t) f(t^2)\, dt$. Find $F'(2)$. [10 pts]
   **Hint:** We can't differentiate with the $x$ inside. How can we get it outside?

10. Show that the Taylor expansion of $f(x) = \sin(4x)$ around any $x_0$ converges for all $x$. [10 pts]
    **Hint:** Use the theorem from class.

11. Define $f_n : [3, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{1}{nx+1}$. Find the function $f : [3, \infty) \to \mathbb{R}$ to which $\{f_n\}$ converges pointwise and then show this convergence is uniform. [15 pts]
    **Hint:** Straightforward. When you're showing unform convergence make sure to bound $x$ but leave $n$ first.

12. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \frac{1}{x+1}$. Using our proof of the Weierstrass Approximation Theorem find the minimum degree polynomial which approximates $f(x)$ uniformly within $\epsilon = 0.1$ and write this polynomial in $\Sigma$ form. [15 pts]
**Hint:** Finding $\delta$ involves looking at the steepest part of $f$. Any reasonable $\delta$ will be accepted.

13. Use Taylor Polynomials to prove that $\int_0^1 e^{(x^2)}\ dx \geq \frac{4}{3}$. [15 pts]
**Hint:** You'll have to consider which Taylor polynomial to use but don't go too far.

14. Show that $\sum_{k=1}^{\infty} \frac{3}{k^4 5^k}$ converges using the Weierstrass Convergence Criterion. [15 pts]
**Hint:** You'll need to bound the $k$ away in the criterion before making sure $n$ can be chosen for any $\epsilon$. The geometric sum formula will be helpful.