MATH 410 Final Exam Review Worksheet

1. Give an example of a set where the supremum exists but the maximum does not.
2. Show that the supremum of $\{x \mid x^2 < 5\}$ is a solution to $x^2 = 5$.
3. Is the definition $S \subseteq \mathbb{R}$ is dense if for each $x \in \mathbb{R}$ there is some $s \in S$ arbitrarily close to x equivalent to our definition?
4. Give (and prove) an example of an ϵ such that the $\epsilon - N$ definition of limit fails when trying to prove $\lim\limits_{n\to\infty} \frac{n}{n+1} = 2$.
5. If $S \subseteq \mathbb{R}$ does every convergent sequence in S converge in S? If not can you think of any additional conditions on S or the sequence under which it must?
6. If $\{a_n\}$ is a sequence list as many criteria as you can for which the sequence must converge and as many criteria as you can for which the sequence must have a convergent subsequence.
7. If $\{a_n\}$ is monotone and bounded above by 5 must it converge to 5?
8. For which $p \in \mathbb{N}$ does $\{a_n^p\}$ convergent imply $\{a_n\}$ convergent?
9. Given that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ prove that $\lim\limits_{n\to\infty} \frac{5}{4n+3} = 0$ by the Comparison Lemma.
10. Use the limit definition of continuity to show that for $f : \mathbb{R} \to \mathbb{R}$ that if $f(x) > 2$ for $x < 3$ then $f(3) \geq 2$. Identify exactly where your argument fails if $\geq$ is replaced by $>$.
11. Give an example of a function which fails both the continuity and closed and bounded interval requirement for the EVT but still satisfies the result.
12. Explain how the IVT can be used to prove the existence of solutions to equations. Cite an example.
13. Explain how vertical asymptotes ruin uniform continuity and give an example.
14. Give an example of a function $f : [0,1] \to \mathbb{R}$ which is bounded but is not uniformly continuous.
15. Give an example of a function $f : D \to \mathbb{R}$ which is continuous everywhere and differentiable nowhere.
16. Give an example of a function and an x-value for which the derivative at that point can only be found using the limit definition and not using basic calculus rules.
17. Explain how the MVT can be used to bound the number of x-intercepts of a continuous function and give an example.
18. Is it true that for $f, g : \mathbb{R} \to \mathbb{R}$ that $f(x) > g(x) > 0$ for all x implies $f'(x) > g'(x)$ for all x? How about if *for all* is replaced by *for some*?
19. Find some $f : [-1,1] \to \mathbb{R}$ such that $f'(x) = |x|$ using the Second Fundamental Theorem.
20. If $f : [-1,1] \to \mathbb{R}$ is such that $f(0) = a$, $f'(0) = b$ and $|f''(x)| \leq c$ for $x \in [-1,1]$ and constants $a, b, c \in \mathbb{R}$, find a bound on $|f(x)|$ for all $x \in [-1,1]$.
21. If $f : [a,b] \to \mathbb{R}$ is integrable on every interval $[c,d]$ with $a < c < d < b$ must f be integrable on $[a,b]$?
22. For $a, b \in \mathbb{R}$ with $a < b$ give an example of a function $f : [0,1] \to \mathbb{R}$ for which $\underline{\int_0^1} f = a$ and $\overline{\int_0^1} f = b$.
23. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \frac{1}{2^k}$ for $x \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]$ for $k = 0, 1, \ldots$ and $f(0) = 0$. Describe a sequence of partitions which could be used to show that f is integrable on $[0,1]$.
24. Give an example of an integrable function $f : [0,1] \to \mathbb{R}$ with the property that $\int_0^1 f \geq 0$ but it is not true that $f(x) \geq 0$ for all $x \in [0,1]$.

25. For $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x$ use the Second Fundamental Theorem and the regular partition of $[0, x]$ along with upper Darboux sums to find a limit expression for the antiderivative of f. Your limit should be a limit as $n \to \infty$ and will have an x in it.

26. What is $\sum_{k=0}^{\infty} (-1)^k$? Prove it.

27. Suppose the fourth Taylor polynomial for $f(x)$ at $x_0 = 0$ is $P_4(x) = 2x^4 - x^2 + 5x + 10$. Find $f^{(k)}(x_0)$ for $k = 0, 1, 2, 3, 4$. How could you do this if x_0 is replaced by $x_0 = 1$?

28. Using the Lagrange Remainder Formula which Taylor polynomial for $f(x) = \sin(2x)$ at $x_0 = 0$ would be necessary to guarantee that the approximation for $\sin(0.5)$ were accurate to within 0.01?

29. If Theorem 8.14 were rewritten with M^n replaced by a more general sequence $\{a_n\}$ what condition on $\{a_n\}$ would still guarantee the general idea of the theorem? Can you give an example of a sequence which satisfies this new theorem but not Theorem 8.14?

30. Use the Weierstrass Approximation Theorem with $f(x) = x^2$ and $n = 2$ to construct a polynomial approximation to f which is not f itself.

31. Give an example of a subset $S \subseteq \mathbb{R}$ and a Cauchy sequence in S which does not converge in S. Does this contradict our theorem that a sequence is Cauchy iff it converges?

32. In the proof of the Weierstrass Approximation Theorem explain intuitively why a smaller value of M should lead to a lower degree polynomial. Furthermore find a bound on M in terms of δ and ϵ which would guarantee that a linear polyomial would suffice.

33. For a series $\sum_{k=0}^{\infty} a_k$ explain what significant difference there is between $a_k = \frac{1}{k}$ and $s_n = \frac{1}{n}$ where s_n is the n^{th} partial sum.

34. Explain why e^x cannot be uniformly approximated by polynomials on all of $\mathbb{R}$. Can you think of some other functions which either can or cannot be uniformly approximated by polynomials on all of $\mathbb{R}$?

35. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a solution to the initial value problem $f'(x) = x^2 + xf(x)$ with $f(1) = 2$. Find the fourth Taylor polynomial at $x_0 = 1$ which approximates the solution.

36. Suppose $f : (-1, 1) \to \mathbb{R}$ has the property that $f^{(n)}(x) \leq n!$ for all $x \in (-1, 1)$. Show that f equals its Taylor Expansion at $x = 0$.

37. Suppose $\{f_n : [0, 1] \to \mathbb{R}\}$ converges uniformly on $[a, 1]$ for all $a > 0$. Must it converge uniformly on $[0, 1]$? What if *uniformly* is replaced by *pointwise*?

38. Construct a sequence of nonintegrable functions $\{f_n : [0, 1] \to \mathbb{R}\}$ which converges uniformly to an integrable function.

39. If $f : [a, b] \to \mathbb{R}$ is continuous describe a visual process by which you could construct a sequence $\{f_n : [a, b] \to \mathbb{R}\}$ of continuous functions which converge pointwise but not uniformly to f.

40. Could a sequence of bounded functions converge uniformly to an unbounded function? How about pointwise? How about if the sequence function are unbounded and the target is bounded?

41. Modify an example given in class to construct a sequence $\{f_n : [0, 1] \to \mathbb{R}\}$ converging to some $f : [0, 1] \to \mathbb{R}$ such that $\lim_{n\to\infty} \int_0^1 f_n = \infty$ but $\int_0^1 f = 0$.

42. Give an example of a sequence $\{f_n : (-1, 1) \to \mathbb{R}\}$ that converges uniformly but for which the sequence $\{f_n'(0)\}$ is unbounded. If you can't come up with an explicit function try to come up with graphs.