MATH 410: Homework 11 **Due in class Monday 12/3/2012**

1. Suppose $a, b \in \mathbb{R}^+$. Prove that the Taylor expansion of $f(x) = a^{bx}$ around any x_0 converges for all x. **

2. Let $x_0 \in \mathbb{R}^+$ and define $f : \mathbb{R} - \{0\} \to \mathbb{R}$ by $f(x) = \frac{1}{x}$.

   (a) Find $p_n(x)$ around x_0. Use the geometric sum formula **

   $$\sum_{k=0}^{n} z^k = \frac{1 - z^{k+1}}{1 - z}$$

   to simplify $p_n(x)$ as much as possible.

   (b) Show directly that for every $n \in \mathbb{N}$ we have *

   $$f(x) - p_n(x) = \frac{\left(1 - \frac{x}{x_0}\right)^{n+1}}{x}$$

   (c) Use part (b) to prove that $f(x)$ equals its Taylor Expansion iff $x \in (0, 2x_0)$. **

3. For each of the following write down and expression for the remainder using both the Lagrange remainder and the Cauchy remainder formulas. For the former clarify what value c could be. Simplify as much as possible (don't integrate the latter).

   (a) $f(x) = \sqrt{x}$, $x_0 = 4$, $R_4(4.5)$. *

   (b) $f(x) = \cos(x)$, $x_0 = \pi$, $R_7(3)$. *

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ has derivatives of all order and all are continuous. Furthermore suppose $\{a_n\}$ is a bounded sequence and f has the property that $|f^{(n)}(x)| \leq a_n$ for all x. Use the Cauchy Remainder Formula to show that f equals its Taylor Expansion. **

5. Use the result of problem 4 from section 8.5 of the book to show that the CRT implies the LRT if $f^{(n+1)} : I \to \mathbb{R}$ is assumed to be continuous. That is, prove that ***

   $$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n \ dt$$

   implies

   $$R_n(x) = \tfrac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1} \text{ for some } c_x \text{ between } x_0 \text{ and } x$$

6. Show that the Approximation Theorem does not hold if we replace I by a bounded open interval (a, b) by showing that if $f(x) = \frac{1}{b-x}$ for all x then $f : (a, b) \to \mathbb{R}$ cannot be uniformly approximated by polynomials. **

7. Find the minimum degree of the polynomial constructed via the Approximation Theorem for each of the following functions defined on $[0, 1]$ with $\epsilon = 0.1$.

   (a) $f(x) = 2\left|x - \frac{1}{2}\right|$ *

   (b) $f(x) = 2^x$ **