Discrete Dynamical Systems

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17.1 Introduction

We solve linear discrete dynamical systems using diagonalization. Specific examples include predator-prey systems and recurrence relations such as the Fibonacci sequence.

17.2 A Predator-Prey System

Suppose there is a population of owls (the predators) living among a population of rats (the prey). A significant portion of the owls' diet consists of the rats. At time goes on, we keep track of the number of owls as well as the number of rats. Let $\emptyset_k$ denote the number of owls after k months, and let R_k denote the number of rats (in thousands) after k months. Our main assumption is that the sizes of these populations evolve according to the following equations:

$$\begin{cases} \emptyset_{k+1} = (0.4)\emptyset_k + (0.5)R_k \\ R_{k+1} = -(0.2)\emptyset_k + (1.2)R_k \end{cases}$$

If we know the populations in a given month, we can predict what the populations will be in the following month. To understand these equations, it helps to consider the extreme cases. For example, consider the owl equation and suppose that there are no rats. Then the equation says $\emptyset_{k+1} = (0.4)\emptyset_k$, which tells us that all but 40% of the owls died as the month passed. This stands to reason, because they have no food. On the other hand, if R_k is very large, then the owl population $\emptyset_{k+1}$ will grow to be larger than $\emptyset_k$.

For the rats, consider what happens if there are no owls. We would have $R_{k+1} = (1.2)R_k$, which indicates that the rat population grows (in fact exponentially, which means at a rate proportional to its own size). But owls exist, and their hunting of rats certainly will take away from the population of rats, as indicated in the equation. If there are too many owls, perhaps they would take eat rats too rapidly, to the point that the rat population was unsustainable. And if the rats disappear, we know what happens to the owls...

So it is intereting to consider what will happen to the populations as time goes on. We'd like to keep track of the two quantities $\emptyset_k$ and R_k, so we'll do so in the vector $\bar{x}_k = \begin{bmatrix} \emptyset_k \\ R_k \end{bmatrix}$. The given system of equations implies

$$\bar{x}_{k+1} = \begin{bmatrix} \emptyset_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} (0.4)\emptyset_k + (0.5)R_k \\ -(0.2)\emptyset_k + (1.2)R_k \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} \emptyset_k \\ R_k \end{bmatrix} = A\bar{x}_k$$

where $A = \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix}$. The equation $\bar{x}_{k+1} = A\bar{x}_k$ that we have obtained is called a *linear discrete dynamical system*. It is discrete because we are keeping track of time in integer increments (as opposed to having time be a continuous variable).

To give us something to solve for, we need an *initial condition*, values of our population at time $k = 0$. Suppose that there are initially 500 owls and 250 (thousand) rats. In our notation,

$$\bar{x}_0 = \begin{bmatrix} \O_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 500 \\ 250 \end{bmatrix}.$$

To summarize so far, we have

$$\bar{x}_{k+1} = \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix} \bar{x}_k, \qquad \bar{x}_0 = \begin{bmatrix} 500 \\ 250 \end{bmatrix},$$

and our goal is to understand the behavior of $\bar{x}_k$ as k increases. The ideal solution for us would be to be able to give an explicit formula for $\bar{x}_k$.

Let's start experimenting to see how many owls and rats there are as time goes on. We'll compute $\bar{x}_1, \bar{x}_2, \bar{x}_3$, etc. Each vector is obtained from the previous one by multiplying by the given 2×2 matrix, which we call A.

$$\begin{aligned}
\bar{x}_1 = A\bar{x}_0 &= \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} 500 \\ 250 \end{bmatrix} = \begin{bmatrix} 325 \\ 200 \end{bmatrix} \\
\bar{x}_2 = A\bar{x}_1 &= \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} 325 \\ 200 \end{bmatrix} = \begin{bmatrix} 230 \\ 175 \end{bmatrix} \\
\bar{x}_3 = A\bar{x}_2 &= \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} 230 \\ 175 \end{bmatrix} = \begin{bmatrix} 179.5 \\ 164 \end{bmatrix} \\
\bar{x}_4 = A\bar{x}_3 &= \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} 179.5 \\ 164 \end{bmatrix} = \begin{bmatrix} 153.8 \\ 160.9 \end{bmatrix} \\
&\vdots
\end{aligned}$$

This looks grim. Both populations appear to be descreasing. Will they both go to zero? Will they stabilize at certain values? Will they bounce back and grow to infinity? We can't be sure until we investigate further.

Notice that each time we compute then next $\bar{x}_{k+1}$, we are multiplying the previous $\bar{x}_k$ by A (in fact, that's just a restatement of what the dynamical system is.) But this means that

$$\begin{aligned}
\bar{x}_1 &= A\bar{x}_0 \\
\bar{x}_2 &= A\bar{x}_1 = A^2\bar{x}_0 \\
\bar{x}_3 &= A\bar{x}_2 = A^3\bar{x}_0 \\
&\vdots
\end{aligned}$$

and in general we have $\bar{x}_k = A^k\bar{x}_0$.

Theorem 17.2.0.1. The solution of the linear discrete dynamical system $\bar{x}_{k+1} = A\bar{x}_k$ is $\bar{x}_k = A^k \bar{x}_0$.

This is great, but if we want to give a formula for $\bar{x}_k$, we are going to need a formula for the k-th power A^k.

17.3 Diagonalization to the Rescue!

Matrix multiplication is a sufficiently strange operation that it is not so clear what A^k will look like on the surface. However, it is pretty clear in the case where the matrix is diagonal.

Example 17.1. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Let's compute powers of D:

$$\begin{aligned}
D^2 &= DD = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix} \\
D^3 &= DD^2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix} \\
D^4 &= DD^3 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix} = \begin{bmatrix} 2^4 & 0 \\ 0 & 3^4 \end{bmatrix} \\
&\vdots
\end{aligned}$$

It becomes apparent that $D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$ for any positive integer k.

Theorem 17.3.0.1. Let $D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$ be an $n \times n$ diagonal matrix. Then for any positive integer k, we have $D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$

The philosophy behind diagonalization is that a diagonalizable matrix is not so different from a diagonal matrix. This is somewhat striking because diagonal matrices are very special, whereas most (is a sense that can be made precise) matrices are diagonalizable. Once we figure out how to do something for a diagonal matrix, we can hope to use that to do the same thing to a diagonalizable matrix. We'll carry this out for powers of a matrix.

Suppose now that $A = PDP^{-1}$. We'll compute powers of A, constantly using the fact that $P^{-1}P = I$.

$$\begin{aligned}
A^2 &= AA = (PDP^{-1})(PDP^{-1}) = PDIDP^{-1} = PD^2P^{-1} \\
A^3 &= AA^2 = (PDP^{-1})(PD^2P^{-1}) = PDID^2P^{-1} = PD^3P^{-1} \\
A^4 &= AA^3 = (PDP^{-1})(PD^3P^{-1}) = PDID^3P^{-1} = PD^4P^{-1} \\
&\ \ \vdots
\end{aligned}$$

It becomes clear that $A^k = PD^kP^{-1}$.

Theorem 17.3.0.2. Suppose that $A = PDP^{-1}$. Then $A^k = PD^kP^{-1}$ for any positive integer k.

Example 17.2. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, which can be diagonalized as

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} = PDP^{-1}.$$

We can compute $A^k = PD^kP^{-1}$ now, since we know how to compute D^k. We have

$$\begin{aligned}
A^k &= PD^kP^{-1} \\
&= \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2^k & (1/2)3^k \\ 2^k & 3^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2^{k+1} - 3^k & -2^k + 3^k \\ 2^{k+1} - 2 \cdot 3^k & -2^k + 2 \cdot 3^k \end{bmatrix}.
\end{aligned}$$

This is our formula for A^k. For example, when $k = 4$ we have

$$A^4 = \begin{bmatrix} 2^5 - 3^4 & -2^4 + 3^4 \\ 2^5 - 2 \cdot 3^4 & -2^4 + 2 \cdot 3^4 \end{bmatrix} = \begin{bmatrix} 32 - 81 & -16 + 81 \\ 32 - 2 \cdot 81 & -16 + 2 \cdot 81 \end{bmatrix} = \begin{bmatrix} -49 & 65 \\ -130 & 146 \end{bmatrix},$$

which can be confirmed by direct matrix multiplication. Notice that the entries of A^k are growing exponentially, in this case there are exponential terms with base 2 and 3. The numbers 2 and 3 are special because they are the eigenvalues of A. The eigenvalues dictate the rate of exponential growth (or decay) of the powers A^k.

Now we can return to our predator-prey system

$$\bar{x}_{k+1} = \begin{bmatrix} 0.4 & 0.5 \\ -0.2 & 1.2 \end{bmatrix} \bar{x}_k, \qquad \bar{x}_0 = \begin{bmatrix} 500 \\ 250 \end{bmatrix}.$$

We know that the solution has the form

$$\bar{x}_k = A^k \bar{x}_0.$$

We need to diagonalize A to compute A^k. Using MATLAB, we see that the eigenvalues for A are (approximately) $\lambda_1 = 1.045$ and $\lambda_2 = 0.555$ with eigenvectors

$$\bar{v}_1 = \begin{bmatrix} 0.613 \\ 0.790 \end{bmatrix} \quad \text{and} \quad \bar{v}_2 = \begin{bmatrix} 0.955 \\ 0.296 \end{bmatrix}.$$

So A can be diagonalized as

$$A = PDP^{-1} = \begin{bmatrix} 0.613 & 0.955 \\ 0.790 & 0.296 \end{bmatrix} \begin{bmatrix} 1.045 & 0 \\ 0 & 0.555 \end{bmatrix} \begin{bmatrix} -0.517 & 1.666 \\ 1.378 & -1.069 \end{bmatrix}.$$

Now if we compute A^k, we get

$$A^k = PD^kP^{-1} = \begin{bmatrix} 0.613 & 0.955 \\ 0.790 & 0.296 \end{bmatrix} \begin{bmatrix} (1.045)^k & 0 \\ 0 & (0.555)^k \end{bmatrix} \begin{bmatrix} -0.517 & 1.666 \\ 1.378 & -1.069 \end{bmatrix},$$

which simplifies to

$$A^k = \begin{bmatrix} -0.316(1.045)^k + 1.316(0.555)^k & 1.021(1.045)^k - 1.021(0.555)^k \\ -0.408(1.045)^k + 0.408(0.555)^k & 1.316(1.045)^k - 0.316(0.555)^k \end{bmatrix}.$$

Finally, we use this to compute

$$\bar{x}_k = A^k \bar{x}_0 = A^k \begin{bmatrix} 500 \\ 250 \end{bmatrix} = \begin{bmatrix} 96.91(1.045)^k + 403.1(0.555)^k \\ 125.0(1.045)^k + 125.0(0.555)^k \end{bmatrix}.$$

Quite explicitly, we now have the solution

$$\begin{cases} \emptyset_k = 96.91(1.045)^k + 403.1(0.555)^k \\ R_k = 125.0(1.045)^k + 125.0(0.555)^k. \end{cases}$$

There is some positive news in here! Despite the initial dip in population, it looks like our animal friends will thrive after all. To see it a little better, it helps to observe that $(0.555)^k \approx 0$ for large k. So for large k, we have

$$\begin{cases} \emptyset_k \approx 96.91(1.045)^k \\ R_k \approx 125.0(1.045)^k. \end{cases}$$

In the long run, both populations will ultimately grow exponentially. The eigenvalue 0.555 did not play much of a role, but the fact that the other eigenvalue

1.045 was greater than 1 was crucial. After a long time, the proportion of rats to owls tends to

$$\frac{R_k}{\emptyset_k} \approx \frac{125.0}{96.91} \approx 1.290$$

Suppose we slightly alter the original dynamical system to become

$$\bar{x}_{k+1} = \begin{bmatrix} 0.4 & 0.5 \\ -0.25 & 1.2 \end{bmatrix} \bar{x}_k.$$

Only one entry changed, but the interpretation is that the owls need to eat more rats to survive (maybe its a different species of owl). We still have $\bar{x}_k = A^k \bar{x}_0$. When we go to diagonalize A, we see that its eigenvalues are now $\lambda_1 = 0.987$ and $\lambda_2 = 0.613$. Each has an eigenvector (they changed as well), but won't give them right now. The matrix A is diagonalizable, and when we compute A^k, we'll get

$$A^k = PD^kP^{-1} = P \begin{bmatrix} (0.987)^k & 0 \\ 0 & (0.613)^k \end{bmatrix} P^{-1}.$$

Since the eigenvalues are both between 0 and 1, we see that

$$\begin{aligned} \lim_{k\to\infty} A^k &= P \begin{bmatrix} \lim_{k\to\infty}(0.987)^k & 0 \\ 0 & \lim_{k\to\infty}(0.613)^k \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

So for *any* initial condition $\bar{x}_0$, we see

$$\lim_{k\to\infty} \bar{x}_k = \lim_{k\to\infty} A^k \bar{x}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \bar{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both species die out regardless of the initial population sizes. It seems that these owls and rats are incompatible.

17.4 Recurrence Relations

Next we'll consider recurrence relations, and we'll show how they give rise to discrete dynamical systems. The classic example is the sequence of *Fibonacci numbers*. We first define the 0-th and 1-st Fibonacci numbers to be $F_0 = 0$ and $F_1 = 1$. For $n \geq 2$, the n-th Fibonacci number is defined recursively as

$$F_n = F_{n-1} + F_{n-2}.$$

That is, the next number in the Fibonacci sequence is the sum of the previous two Fibonacci numbers. So we can compute

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= F_1 + F_0 = 1 + 0 = 1 \\
F_3 &= F_2 + F_1 = 1 + 1 = 2 \\
F_4 &= F_3 + F_2 = 2 + 1 = 3 \\
F_5 &= F_4 + F_3 = 3 + 2 = 5 \\
F_6 &= F_5 + F_4 = 5 + 3 = 8 \\
F_7 &= F_6 + F_5 = 8 + 5 = 13 \\
F_8 &= F_7 + F_6 = 13 + 8 = 21 \\
&\vdots
\end{aligned}$$

We can keep computing Fibonacci numbers until we get bored:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

It seems that the sequence eventually starts growing very rapidly. But how rapidly? Does it grow linearly? quadraticly? exponentially?

Here is another question: what is F_{100}? It seems that the only way to compute it would be to compute every single Fibonacci number before F_{100}, and then obtain F_{100} as the sum $F_{99} + F_{98}$. Maybe that's doable, but what if we want F_{1000} or F_{10000}? We will answer all of these questions by finding a closed formula for F_k in terms of k alone.

If we know one Fibonacci number F_k, we do not have sufficient information to determine the next Fibonacci number F_{k+1}. However, we will have enough information if we know a *pair* of consecutive Fibonacci numbers. In fact, if we know a pair (F_k, F_{k+1}) of consecutive Fibonacci numbers, we can determine the next pair (F_{k+1}, F_{k+2}), since F_{k+2} is the sum of the pair that we know.

This observation motivates us to consider pairs of consecutive Fibonacci numbers, which we keep track of in a vector. Let

$$\bar{x}_k = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}.$$

Then the vector $\bar{x}_{k+1}$ can be obtained from $\bar{x}_k$:

$$\bar{x}_{k+1} = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k + F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \bar{x}_k.$$

We've obtained a discrete dynamical system, and the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is playing a major role. More specifically, we have

$$\bar{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \bar{x}_k, \qquad \bar{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Note that the initial vector $\bar{x}_0$ just contains the first two Fibonacci numbers. If we can find a formula for $\bar{x}_k$, then we will have a formula for F_k. We know how to do this now:

$$\bar{x}_k = A^k \bar{x}_0.$$

We must diagonalize the matrix A so that we can simplify A^k. Its eigenvalues are found by solving

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
\det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} &= 0 \\
\lambda^2 - \lambda - 1 &= 0 \\
\lambda &= \frac{1 \pm \sqrt{5}}{2},
\end{aligned}$$

where the solutions were obtained by using the quadratic formula. We have two eigenvalues, which we will denote as

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1-\sqrt{5}}{2}.$$

Corresponding eigenvectors can be found as

$$\bar{v}_\varphi = \begin{bmatrix} 1 \\ \varphi \end{bmatrix} \quad \text{and} \quad \bar{v}_\psi = \begin{bmatrix} 1 \\ \psi \end{bmatrix}.$$

So we can diagonalize $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ \varphi & \psi \end{bmatrix} \quad \text{and } D = \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix}.$$

We'll need P^{-1} also. Note that $\det P = \psi - \varphi = -\sqrt{5}$. So

$$P^{-1} = \frac{1}{\det P} \begin{bmatrix} \psi & -1 \\ -\varphi & 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix}.$$

Our diagonalization is

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ \varphi & \psi \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix}.$$

We need A^k, which is

$$\begin{aligned}
A^k &= PD^kP^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ \varphi & \psi \end{bmatrix} \begin{bmatrix} \varphi^k & 0 \\ 0 & \psi^k \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix} \\
&= \begin{bmatrix} \varphi^k & \psi^k \\ \varphi^{k+1} & \psi^{k+1} \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi & 1 \\ \varphi & -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi\varphi^k + \varphi\psi^k & \varphi^k - \psi^k \\ -\psi\varphi^{k+1} + \varphi\psi^{k+1} & \varphi^{k+1} - \psi^{k+1} \end{bmatrix}.
\end{aligned}$$

Finally, we have

$$\begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = A^k \bar{x}_0 = \frac{1}{\sqrt{5}} \begin{bmatrix} -\psi\varphi^k + \varphi\psi^k & \varphi^k - \psi^k \\ -\psi\varphi^{k+1} + \varphi\psi^{k+1} & \varphi^{k+1} - \psi^{k+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^k - \psi^k \\ \varphi^{k+1} - \psi^{k+1} \end{bmatrix}.$$

The top entry of this vector gives our desired formula

$$F_k = \frac{1}{\sqrt{5}}(\varphi^k - \psi^k) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k + \left(\frac{1-\sqrt{5}}{2} \right)^k \right).$$

It seems remarkable already that this expression is always an integer. To understand it a little better, observe that

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$
$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.618.$$

In particular, we see that $\psi^k \approx 0$ for large k. This means that the k-th Fibonacci number is approximately

$$F_k \approx \frac{\varphi^k}{\sqrt{5}}.$$

This can be used to compute Fibonacci numbers. For example,

$$F_{21} \approx \frac{\varphi^{21}}{\sqrt{5}} \approx 10945.99998.$$

To get the Fibonacci number, we just round to the nearest integer: $F_{21} = 10946$.

Our formula for F_k also indicates that the Fibonacci sequence grows exponentially, with base $\varphi \approx 1.618$. This number φ is called the *golden ratio*. The golden ratio is, approximately, the ratio of consecutive Fibonacci numbers:

$$\frac{F_{k+1}}{F_k} \approx \frac{\frac{1}{\sqrt{5}}\varphi^{k+1}}{\frac{1}{\sqrt{5}}\varphi^k} = \varphi.$$

More precisely, φ is the asymptotic ratio of consecutive Fibonacci numbers, meaning

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \varphi.$$