1. State the following three definitions:
   (a) Define what it means for $\{x_n\} \to x_0$.
   Solution: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - x_0| < \epsilon$
   (b) Define what it means for a set $S \subseteq \mathbb{R}$ to be closed.
   Solution: Every convergent sequence in S converges to something in S.
   (c) Define what it means for a function $f : D \to \mathbb{R}$ to be uniformly continuous.
   Solution: If $\{u_n\}$ and $\{v_n\}$ are in D and $\{u_n - v_n\} \to 0$ then $\{f(u_n) - f(v_n)\} \to 0$.

2. State the Intermediate Value Theorem. Pick one hypothesis, remove it, and give a counterexample showing the new statement is false.
   Solution: If $f : [a, b] \to \mathbb{R}$ is continuous and c is strictly between $f(a)$ and $f(b)$ then there exists some $x_0 \in (a, b)$ with $f(x_0) = c$.

   One option: If the continuity hypothesis is removed then $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$ is a counterexample with $c = 0.5$.

3. The following is true for any convergent sequence $\{x_n\} \to x_0$:

   $$\text{If } x_0 > 0 \text{ then } \exists N \in \mathbb{N}, \forall n \geq N, x_n > 0.$$

   State the converse and give a counterexample showing that the converse is false.
   Solution: The converse is

   $$\text{If } \exists N \in \mathbb{N}, \forall n \geq N, x_n > 0 \text{ then } x_0 > 0.$$

   A counter example is $\{1/n\}$ for which all terms are all greater than 0 and yet it converges to 0.

4. Prove using ϵ-N that:

   $$\left\{2 - \frac{1}{n} + \frac{3}{n^2}\right\} \to 2$$

   Solution: For scratch we want to choose N so $n \geq N$ implies

   $$\left|2 - \frac{1}{n} + \frac{3}{n^2} - 2\right| < \epsilon$$

   Observe that

   $$\left|2 - \frac{1}{n} + \frac{3}{n^2} - 2\right| = \left|-\frac{1}{n} + \frac{3}{n^2}\right| \leq \left|-\frac{1}{n}\right| + \left|\frac{3}{n^2}\right| = \frac{1}{n} + \frac{3}{n^2} \leq \frac{1}{n} + \frac{3}{n} = \frac{4}{n}$$

   so if $\frac{4}{n} < \epsilon$ or $n > \frac{\epsilon}{4}$ we are safe.

   To be formal let ϵ be given and choose $N > \frac{\epsilon}{4}$. Then if $n \geq N$ then $n > \frac{\epsilon}{4}$ and so $\frac{4}{n} < \epsilon$ and then

   $$\left|2 - \frac{1}{n} + \frac{3}{n^2} - 2\right| = \left|-\frac{1}{n} + \frac{3}{n^2}\right| \leq \left|-\frac{1}{n}\right| + \left|\frac{3}{n^2}\right| = \frac{1}{n} + \frac{3}{n^2} \leq \frac{1}{n} + \frac{3}{n} = \frac{4}{n} < \epsilon$$

   as desired.

5. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 2x & \text{if } x \leq 5 \\ 0 & \text{if } x > 5 \end{cases}$$

   Prove using the sequence definition of continuity that $f(x)$ is continuous at $x = 3$.
   Solution: Suppose $\{x_n\} \to 3$. Choose N so that if $n \geq N$ then $|x_n - 3| < 2$. Then $x_n < 5$ and so $\{f(x_n)\} = \{2x_n\} \to 2(3) = 6 = f(3)$.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x_0 \in \mathbb{R}$ with $f(x_0) > 0$. Show that there exists some $\alpha > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \alpha, x_0 + \alpha)$
   Solution: By the ϵ-δ criterion with $\delta = f(x_0)/2$ we can choose $\epsilon > 0$ so that if $|x - x_0| < \epsilon$ then $|f(x) - f(x_0)| < f(x_0)/2$. Let $\alpha = \epsilon$ and then if $|x - x_0| < \alpha$ then $|f(x) - f(x_0)| < f(x_0)/2$ which implies that $-f(x_0)/2 < f(x) - f(x_0) < f(x_0)/2$ which implies that $f(x) > f(x_0)/2 > 0$ as desired.

7. Suppose D is sequentially compact and $f : D \to \mathbb{R}$ is continuous. Prove that $f(D)$ is sequentially compact.
   Solution: Suppose $\{y_n\}$ is a sequence in $f(D)$. We claim there is a subsequence converging to something in $f(D)$. Well for all n we have some $x_n \in D$ with $f(x_n) = y_n$. Then $\{x_n\}$ is a sequence in D which by sequential compactness has a subsequence $\{x_{n_k}\} \to x_0 \in D$. Then by continuity $\{f(x_{n_k})\} \to f(x_0) \in f(D)$. Let $y_0 = f(x_0)$ and so $\{f(x_{n_k})\} = \{y_{n_k}\}$ is a subsequence of $\{y_n\}$ which converges to $f(x_0) = y_0 \in f(D)$.

8. Suppose $\{x_n\}$ is a bounded sequence which has the property that for all $n \in \mathbb{N}$ there is some $n_1 > n$ with $x_{n_1} > x_n$ and some $n_2 > n$ with $x_{n_2} < x_n$. Prove that $\{x_n\}$ does not converge.
   Solution: By way of contradiction suppose $\{x_n\} \to x_0$. The hypothesis allows us to construct a monotone increasing subsequence of x_n and a monotone decreasing subsequence of x_n which, since they are bounded, must converge by the MCT to the inf and sup of the set of sequence values. But they must also conver to x_0 which means this inf and sup must both equal x_0 so that the sequence must be constant which contradicts the hypotheses.