1. State the following three definitions:
   (a) Define what it means for $\{x_n\} \to x_0$.
   Solution: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - x_0| < \epsilon$
   (b) Define what it means for a point $x_0 \in D \subseteq \mathbb{R}$ to be a limit point.
   Solution: There exists a sequence in $D - \{x_0\}$ which converges to x_0.
   (c) Define what it means for a function $f : D \to \mathbb{R}$ to be continuous.
   Solution: For all $x_0 \in D$ and for all sequences $\{x_n\}$ in D with $\{x_n\} \to x_0$ we have $\{f(x_n)\} \to f(x_0)$.

2. State the Extreme Value Theorem. Pick one hypothesis, remove it, and give a counterexample showing the new statement is false.
   Solution: The EVT states that if D is closed and bounded and $f : D \to \mathbb{R}$ is continuous then f achieves a max and min value.

   One option: If the hypothesis that D be bounded is removed then $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ has no maximum.

3. The following is true for any monotonically increasing sequence $\{x_n\}$ and any $M \in \mathbb{R}$:

$$\text{If } \{x_n\} \to M \text{ then } \forall n \in \mathbb{N}, x_n \leq M.$$

   State the converse and give a counterexample showing that the converse is false.
   Solution: The converse is

$$\text{If } \forall n \in \mathbb{N}, x_n \leq M \text{ then } \{x_n\} \to M.$$

   A counter example is $\{1 - 1/n\}$ with $M = 2$.

4. Prove using ϵ-N that:

$$\left\{2 - \frac{1}{n} + \frac{3}{n^2}\right\} \to 2$$

   Solution: For scratch we want to choose N so $n \geq N$ implies

$$\left|2 - \frac{1}{n} + \frac{3}{n^2} - 2\right| < \epsilon$$

   Observe that

$$\left|2 - \frac{1}{n} + \frac{3}{n^2} - 2\right| = \left|-\frac{1}{n} + \frac{3}{n^2}\right| \leq \left|-\frac{1}{n}\right| + \left|\frac{3}{n^2}\right| = \frac{1}{n} + \frac{3}{n^2} \leq \frac{1}{n} + \frac{3}{n} = \frac{4}{n}$$

   so if $\frac{4}{n} < \epsilon$ or $n > \frac{\epsilon}{4}$ we are safe.

   To be formal let ϵ be given and choose $N > \frac{\epsilon}{4}$. Then if $n \geq N$ then $n > \frac{\epsilon}{4}$ and so $\frac{4}{n} < \epsilon$ and then

$$\left|2 - \frac{1}{n} + \frac{3}{n^2} - 2\right| = \left|-\frac{1}{n} + \frac{3}{n^2}\right| \leq \left|-\frac{1}{n}\right| + \left|\frac{3}{n^2}\right| = \frac{1}{n} + \frac{3}{n^2} \leq \frac{1}{n} + \frac{3}{n} = \frac{4}{n} < \epsilon$$

   as desired.

5. Consider $f : [3, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x-1}$. Prove from the definition that f is uniformly continuous.
Solution: Suppose $\{u_n\}$ and $\{v_n\}$ are in $[0, \infty)$ such that $\{u_n - v_n\} \to 0$. Observe that:

$$|f(u_n) - f(v_n)| = \left|\frac{1}{u_n - 1} - \frac{1}{v_n - 1}\right| = \left|\frac{v_n - u_n}{(u_n - 1)(v_n - 1)}\right|$$

Since $u_n, v_n \in [3, \infty)$ we have $u_n, v_n \geq 3$ and so $(u_n - 1)(v_n - 1) \leq 4$ and so

$$\left|\frac{v_n - u_n}{(u_n - 1)(v_n - 1)}\right| \leq \frac{1}{4}|v_n - u_n|$$

so that $\{f(u_n) - f(v_n)\} \to 0$ by the Comparison Lemma.

6. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and for all $\epsilon > 0$ there is some $x \in [a, b]$ with $|f(x) - 17| < \epsilon$. Prove there is some $x_0 \in [a, b]$ with $f(x_0) = 17$.
Solution: For each $n \in \mathbb{N}$ choose x_n so that $|f(x_n) - 17| < \frac{1}{n}$ so that $\{f(x_n)\} \to 17$ by the Comparison Lemma. By the sequential compactness of $[a, b]$ the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\} \to x_0 \in [a, b]$. By continuity then $\{f(x_{n_i})\} \to f(x_0)$ but since $\{f(x_{n_i})\} \to 17$ we have $f(x_0) = 17$.

   Alternate Solution: By the EVT the function f has a maximum and a minimum. If either of these is 17 then we're done because the max and min are achieved by the MVT. We cannot have $17 < min \leq max$ since then the hypothesis would fail with $\epsilon = \frac{min-17}{2}$. We cannot have $min \leq max < 17$ since then the hypothesis would fail with $\epsilon = \frac{17-max}{2}$. We must therefore have $min < 17 < max$ in which case a solution exists by the IVT.

7. Problem removed due to error.

8. Suppose $\{x_n\}$ is a strictly decreasing sequence which has a convergent subsequence. Prove that $\{x_n\}$ converges.
Solution: Suppose the convergent subsequence is $\{x_{n_i}\}$ which is bounded below since it is monotone decreasing and converges. Call the bound M so $x_{n_i} \geq M$ for all n_i. We claim $\{x_n\}$ is also bounded below by M. Suppose not, then there is some n with $x_n < M$. Let x_{n_j} be an element later in the subsequence so $n_j > n$ and so $x_{n_j} < x_n < M$ which is a contradiction.