1. State the following three definitions:

   (a) If I is a neighborhood of x_0, define what it means for $f : I \to \mathbb{R}$ to be differentiable at x_0.
   Solution: There exists an L such that for any $\{x_n\}$ in $I - \{x_0\}$ converging to x_0 we have $\left\{\frac{f(x_n)-f(x_0)}{x_n - x_0}\right\} \to L$.

   (b) Given $f : [a, b] \to \mathbb{R}$ and a partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$, define the upper Darboux sum $U(f, P)$.
   Solution: $U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ where M_i is the supremum of f on $[x_{i-1}, x_i]$.

   (c) Define what it means for a bounded function $f : [a, b] \to \mathbb{R}$ to be integrable.
   Solution: f is integrable if $\underline{\int_a^b} f = \overline{\int_a^b} f$.

2. State the Identity Criterion. Pick one hypothesis, remove it, and give a counterexample showing the new statement is false.
   Solution: If I is an open interval and $f, g : I \to \mathbb{R}$ are differentiable then $f' = g'$ iff they differ by a constant. If we remove differentiability then $f, g : (0, 2) \to \mathbb{R}$ defined by $f(x) = 0$ on $(0, 1]$ and $f(x) = 1$ on $(1, 2)$ $g(x) = 0$ on $(0, 1]$ and $g(x) = 2$ on $(1, 2)$ have the same derivative everywhere (that they have one) but they don't differ by a constant.

3. The following is true for any continuous function $f : [a, b] \to \mathbb{R}$:

   $$\text{If } \int_a^b f = 0 \text{ then there is some } x_0 \in [a, b] \text{ with } f(x_0) = 0.$$

   State the converse and give a counterexample showing that the converse is false.
   Solution: The converse is: If there is some $x_0 \in [a, b]$ with $f(x_0) = 0$ then $\int_a^b f = 0$. A counterexample is $f(x) = x^2$ on $[-1, 1]$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 3x$. Use the definition of the derivative and the sequence definition of the limit to find $f'(-2)$.
   Solution: Suppose $\{x_n\}$ is in $\mathbb{R} - \{-2\}$ converging to -2. Then

   $$\left\{\frac{f(x_n) - f(-2)}{x_n - (-2)}\right\} = \left\{\frac{x_n^2 - 3x_n + 10}{x_n + 2}\right\} = \{x_n - 5\} \to -7$$

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $f(3) = 0$, $f'(3) = 1$, $f''(3) = 0$, and $f'''(x) \geq 0.02$ for all x. Use the Function Control Theorem to find a lower bound on $f(3.3)$.
   Solution: Define $g(x) = f(x) - x - 3$ then $g(3) = f(3) - 3 - 3 = f(3) = 0$, $g'(x) = f'(x) - 1$ so $g'(3) = f'(3) - 1 = 1 - 1 = 0$, $g''(x) = f''(x)$ so $g''(3) = 0$ and $g'''(x) = f'''(x) \geq 0.02$ for all x. By the FTC there exists some z between 0 and 0.02 with

   $$g(3.3) = \frac{g'''(z)}{3!}(3.3 - 3)^3 \geq \frac{0.02}{6}(0.3)^3$$

   . Then

   $$f(3.3) = g(3.3) + 3.3 + 3 \geq 6.3 + \frac{0.02}{6}(0.3)^3$$

6. Let $n \in \mathbb{N}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) = 0$ has at most $n - 1$ solutions. Prove that $f(x) = 0$ has at most n solutions.
   Solution: By way of contradiction suppose $f(x) = 0$ has more solutions. Let $x_1 < x_2 < ... < x_{n+1}$ be $n + 1$ of them. For each $i = 1, 2, ..., n$ apply Rolle's Theorem to $[x_i, x_{i+1}]$ to get some $c_i \in (x_i, x_{i+1})$ with $f'(c_i) = 0$. This is a contradiction.

7. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and suppose $G : [a, b] \to \mathbb{R}$ satisfies $G(a) = 0$ and $G'(x) = f(x)$ for all $x \in (a, b)$. Prove $G(x) = \int_a^x f$ for all $x \in (a, b)$.
Solution: Observe that $\frac{d}{dx} \int_a^x f = f(x) = G'(x)$ so that $\int_a^x f$ and $G(x)$ differ by a constant by the Identity Criterion. More specifically there is some C so that for all x we have $\int_a^x f - G(x) = C$ for some C. Then observe that when $x = a$ we have $\int_a^a f - G(a) = C$ and since $\int_a^a f = 0$ and $G(a) = 0$ we have $C = 0$.

8. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and has the property that for all c, d with $a \leq c < d \leq b$ we have $\int_c^d f \geq 0$. Prove that $f(x) \geq 0$ for all $x \in [a, b]$.
Solution: Suppose there exists some $x_0 \in [a, b]$ with $f(x_0) < 0$. Applying ϵ-δ to f at x_0 with $\epsilon = -f(x_0)$ gives us δ such that for $x_0 - \delta < x < x_0 + \delta$ we have $f(x_0) - (-f(x_0)) < f(x) < f(x_0) + (-f(x_0))$ which yields $f(x) < 0$ on that interval. Take $[c, d] = [x_0 - \delta, x_0 + \delta] \cap [a, b]$. Since f is continuous it has a maximum $M < 0$ on $[c, d]$. Then $f(x) < M$ on $[c, d]$ and so $\int_c^d f \leq M(d - c) < 0$, a contradiction.