1. State the following three definitions:

   (a) If I is a neighborhood of x_0, define what it means for $f : I \to \mathbb{R}$ to be differentiable at x_0.
   Solution: There exists an L such that for any $\{x_n\}$ in $I - \{x_0\}$ converging to x_0 we have $\left\{\frac{f(x_n)-f(x_0)}{x_n-x_0}\right\} \to L$.

   (b) Given $f : [a, b] \to \mathbb{R}$ and a partition $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$, define the lower Darboux sum $L(f, P)$.
   Solution: $L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ where m_i is the infimum of f on $[x_{i-1}, x_i]$.

   (c) For a bounded function $f : [a, b] \to \mathbb{R}$ define the upper Darboux integral.
   Solution: $\overline{\int_a^b} f = \inf \{U(f, P) \,|\text{All partitions } P \text{ of } [a, b]\}$

2. State the Mean Value Theorem. Pick one hypothesis, remove it, and give a counterexample showing the new statement is false.
   Solution: Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $x_0 \in (a, b)$ at which $f'(x_0) = \frac{f(b)-f(a)}{b-a}$. If we remove differentiability then a counterexample is $f(x) = |x|$ on $[1, 1]$.

3. For an open interval I the following is true for any differentiable function $f : I \to \mathbb{R}$:

   If $f'(x) < 0$ for all $x \in I$ then f is strictly decreasing.

   State the converse and give a counterexample showing that the converse is false.
   Solution: The converse is: If f is strictly decreasing then $f'(x) < 0$ for all $x \in I$. A counterexample is $f(x) = -x^{1/3}$ on $[-1, 1]$.

4. Define $f : [0, 3] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2 & \text{if } x \in [0, 1] \\ 3 - x & \text{if } x \in (1, 3] \end{cases}$$

   Write down the algebraic rule for an antiderivative of f.
   Solution: Since f is continuous we can apply the SFTOC so $F(x) = \int_0^x f$ is an antiderivative. If $x \in [0, 1]$ then:

$$F(x) = \int_0^x f = \int_0^x 2 = 2t\Big|_0^x = 2x$$

   If $x \in (1, 3]$ then:

$$F(x) = \int_0^x f = \int_0^1 2 + \int_1^x 3 - t = 2x\Big|_0^1 + \left(3t - \frac{1}{2}t^2\right)\Big|_1^x = 2 + \left(3x - \frac{1}{2}x^2\right) - \left(3 - \frac{1}{2}\right)$$

5. Consider $f : [0,2] \to \mathbb{R}$ defined by $f(x) = 2x$. You may assume f is integrable. Use the AR Theorem to calculate $\int_0^2 f$.
Solution: Define $\{P_n\}$ as the sequence of regular partitions so

$$P_n = \left\{0, 1 \cdot \frac{2}{n}, 2 \cdot \frac{2}{n}, ..., (n-1) \cdot \frac{2}{n}, 2\right\}$$

Observe that:

$$L(f, P_n) = \frac{2}{n}\left[2(0) + 2\left(1 \cdot \frac{2}{n}\right) + ... + 2\left((n-1)\frac{2}{n}\right)\right]$$

and

$$U(f, P_n) = \frac{2}{n}\left[2\left(1 \cdot \frac{2}{n}\right) + ... + 2\left((n-1)\frac{2}{n}\right) + 2(2)\right]$$

So that

$$\{U(f, P_n) - L(f, P_n)\} = \left\{\frac{2}{n}[4-0]\right\} \to 0$$

so that $\{P_n\}$ is an ASOP. Then observe that

$$\{U(f, P_n)\} = \left\{\frac{2}{n}\left[\frac{4}{n} + \frac{8}{n} + ... + \frac{4n}{n}\right]\right\} = \left\{\frac{8}{n^2}\frac{n(n+1)}{2}\right\} = \left\{4\left(1 + \frac{1}{n}\right)\right\} \to 4$$

6. Let $x_0 \in \mathbb{R}$. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable and satisfy $f(x_0) < g(x_0)$ and $f'(x) \le g'(x)$ for all $x \ge x_0$. Prove that $f(x) < g(x)$ for all $x \ge x_0$.
Solution: If we define $h(x) = g(x) - f(x)$ then h is differentiable and the problem becomes:
Suppose $h(x)$ satisfies $h(x_0) > 0$ and $h'(x) \ge 0$ for $x \ge x_0$. Prove that $h(x) > 0$ for all $x > x_0$.
Assume by way of contradiction there is some $x_1 \ge x_0$ with $h(x_1) \le 0$. Clearly $x_1 > x_0$ since $h(x_0) > 0$ so then consider by the MVT there is some $x_2 \in (x_0, x_1)$ with:

$$h'(x_1) = \frac{h(x_1) - h(x_0)}{x_1 - x_0} < 0$$

which is a contradiction.

7. Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous. Prove that:

$$\int_a^b |f + g| \le \int_a^b |f| + \int_a^b |g|$$

Solution: This wasn't intended to be so easy but basically by the triangle inequality we have $|f + g| \le |f| + |g|$ and the result follows by the monotonicity of the integral.

8. Suppose $f : [a, b] \to \mathbb{R}$ is monotone increasing. Prove that f is integrable. Of course you may not use the theorem that monotone functions are integrable!
Solution: For the regular partition $\{a = x_0, x_1, ..., x_{n-1}, x_n = b\}$ we have:

$$L(f, P) = \tfrac{b-a}{n}\left[f(x_0) + ... + f(x_{n-1})\right] \qquad \text{and} \qquad U(f, P) = \tfrac{b-a}{n}\left[f(x_1) + ... + f(x_n)\right]$$

It follows that:

$$\{U(f, p) - L(f, P)\} = \left\{\frac{b-a}{n}\left[f(x_n) - f(x_0)\right]\right\} = \left\{\frac{b-a}{n}\left[f(b) - f(a)\right]\right\} \to 0$$

so that f is integrable by the AR Theorem.