Math 410 Section 2.1: Sequences and Convergence

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1. **Definition:** A sequence is formally a function $f : \mathbb{N} \to \mathbb{R}$.
   Example: $f(n) = n^2$.
   We think of a sequence as a succession of terms, though, like if we plugged in $1, 2, 3, ...$ so the above would be $1, 4, 9, 16, ...$
   More typical notation would be one of $a_n = n^2$ or $\{n^2\}$ or giving the terms if it's clear.
   Sometimes a sequence may be given recursively.
   Example: If $a_1 = 4$ and for $n > 1$ we have $a_n = \sqrt{a_{n-1} + 2}$. Then $a_2 = \sqrt{a_1 + 2} = \sqrt{6}$ and $a_3 = \sqrt{a_2 + 2} = \sqrt{\sqrt{6} + 2}$ and so on.
2. **Convergence**
   (a) Idea: We are interested in the long-term behavior of a sequence as $n \to \infty$. For example the terms in the sequence $\left\{\frac{1}{n}\right\}$ approach 0 whle the terms in the sequence $\{2^n\}$ head off to infinity.
   The specific case when the terms in a sequence $\{a_n\}$ approach a specific value a is captured by the following idea that we then formalize:

   "No matter how close we want the terms to get to a, eventually they get that close and stay that close."

   (b) **Definition:** We define $\{a_n\} \to a$ if:

   $$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that if } n \geq N \text{ then } |a_n - a| < \epsilon$$

   In practice when we're using this to show convergence we have to obey the quantifiers - we start with an (unknown) ϵ and show how we can get some N (which will almost always depend on ϵ) so that for $n \geq N$ we have $|a_n - a| < \epsilon$.
   Example:
   Show $\left\{\frac{3}{n}\right\} \to 0$.
   Scratch: Assume $\epsilon > 0$ is given (and unknown). We need N so that if $n \geq N$ then $\left|\frac{3}{n} - 0\right| < \epsilon$.
   Notice that $\left|\frac{3}{n} - 0\right| = \frac{3}{n}$ and $\frac{3}{n} < \epsilon$ iff $n > \frac{3}{\epsilon}$ so as long as $N > \frac{3}{\epsilon}$ we're good.
   Proof: Given $\epsilon > 0$ let $N = \lceil\frac{3}{\epsilon}\rceil + 1$. Then if $n \geq N$ then $n \geq \lceil\frac{3}{\epsilon}\rceil > \frac{3}{\epsilon}$ and so $\left|\frac{3}{n} - 0\right| < \epsilon$.
   Example:
   Show $\left\{\frac{7}{n^2} - \frac{2}{n} + 5\right\} \to 5$
   Scratch: Observe that using the Triangle Inequality

   $$\left|\frac{7}{n^2} - \frac{2}{n} + 5 - 5\right| = \left|\frac{7}{n^2} + \left(-\frac{2}{n}\right)\right| \leq \left|\frac{7}{n^2}\right| + \left|\frac{2}{n}\right| \leq \left|\frac{7}{n}\right| + \left|\frac{2}{n}\right| = \frac{9}{n}$$

   Since $\frac{9}{n} < \epsilon$ iff $n > \frac{9}{\epsilon}$ we know as long as $N > \frac{9}{\epsilon}$ we're good.
   Proof: Given $\epsilon > 0$ let $N = \lceil\frac{9}{\epsilon}\rceil + 1$. Then if $n \geq N$ then $n \geq \lceil\frac{9}{\epsilon}\rceil + 1 > \frac{9}{\epsilon}$ and so

   $$\left|\frac{7}{n^2} - \frac{2}{n} + 5 - 5\right| = \left|\frac{7}{n^2} + \left(-\frac{2}{n}\right)\right| \leq \left|\frac{7}{n^2}\right| + \left|\frac{2}{n}\right| \leq \left|\frac{7}{n}\right| + \left|\frac{2}{n}\right| = \frac{9}{n} < \epsilon$$

Example:

Prove that if $\{a_n\} \to 2$ then $\left\{\frac{1}{a_n}\right\} \to \frac{1}{2}$.

The claim here is that:

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \text{ st if } n \geq N \text{ then } \left|\frac{1}{a_n} - \frac{1}{2}\right| < \epsilon$$

Scratch

So given some unknown $\epsilon > 0$ how can we choose $N \in \mathbb{N}$ so that if $n \geq N$ then $\left|\frac{1}{a_n} - \frac{1}{2}\right| < \epsilon$?

Observe that

$$\left|\frac{1}{a_n} - \frac{1}{2}\right| = \left|\frac{2 - a_n}{2a_n}\right| = \left|\frac{a_n - 2}{2a_n}\right|$$

so really we're trying to make $\left|\frac{a_n - 2}{2a_n}\right|$ small ... less than ϵ.

We know that we can make a_n as close to 2 as we like because $\{a_n\} \to 2$ so we can make the numerator as small as we like but that denominator is awkward. We know it's approaching 4 because $\{a_n\} \to 2$ but we don't have an inequality for it - it could be bigger or smaller than 4 for any given n.

However since $\{a_n\} \to 2$ we know that eventually $a_n > 1$ because eventually $\{a_n\}$ is as close as we like to 2. If $a_n > 1$ then we'd have:

$$\left|\frac{a_n - 2}{2a_n}\right| < \left|\frac{a_n - 2}{2(1)}\right| = \frac{1}{2}|a_n - 2|$$

at this point we can make $|a_n - 2| < 2\epsilon$ and we have what we want.

Note that we need two cutoffs here. We need N_1 beyond which $a_n > 1$ and N_2 beyond which $|a_n - 2| < 2\epsilon$.

Formal Proof

Given $\epsilon > 0$:

- Choose N_1 so that if $n \geq N_1$ then $|a_n - 2| < 1$ so that $a_n > 1$.
- Choose N_2 so that if $n \geq N_2$ then $|a_n - 2| < 2\epsilon$.

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$ then we have:

$$\left|\frac{1}{a_n} - \frac{1}{2}\right| = \left|\frac{2 - a_n}{2a_n}\right| = \left|\frac{a_n - 2}{2a_n}\right| < \left|\frac{a_n - 2}{2(1)}\right| = \frac{1}{2}|a_n - 2| < \frac{1}{2}(2\epsilon) = \epsilon$$

QED

3. **The Comparison Lemma**
   The Comparison Lemma is a very useful tool for showing convergence of one sequence based on convergence of another.
   Theorem (The Comparison Lemma):
   Suppose $\{a_n\} \to a$ and suppose $\{b_n\}$ is a sequence and $b \in \mathbb{R}$. Now suppose there is some $C \in \mathbb{R}^+$ and some $N \in \mathbb{N}$ such that if $n \geq N$ then $|b_n - b| < C|a_n - a|$. Then $\{b_n\} \to b$.
   Intuition: We need to get $|b_n - b|$ small. We go far enough in the sequence for the inequality to be true and far enough for $C|a_n - a|$ to be small enough.
   Proof:
   Let $\epsilon > 0$. Choose N_1 so that if $n \geq N_1$ then $|b_n - b| < C|a_n - a|$. Choose N_2 so that if $n \geq N_2$ then $|a_n - a| < \frac{\epsilon}{C}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$ then

   $$|b_n - b| < C|a_n - a| < C\left(\frac{\epsilon}{C}\right) = \epsilon$$

   $\mathcal{QED}$

4. **Theorem (Combinations):**
   If $\{a_n\} \to a$ and $\{b_n\} \to b$ then:
   (a) $\{a_n \pm b_n\} \to a \pm b$
   Proof for +: Let $\epsilon > 0$. Choose N_1 so that if $n \geq N_1$ then $|a_n - a| < \frac{\epsilon}{2}$ and choose N_2 so that if $n \geq N_2$ then $|b_n - b| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ then if $n \geq N$ then

   $$|(a_n + b_n) - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \epsilon$$

   $\mathcal{QED}$

   (b) $\{a_n b_n\} \to ab$
   (c) $\left\{\frac{a_n}{b_n}\right\} \to \frac{a}{b}$ provided $b \neq 0$ and $\forall n, b_n \neq 0$.
   (d) If $p(x)$ is a polynomial then $\{p(a_n)\} \to p(a)$.