Math 410 Section 2.2: Bounded, Sequential Density, Closed

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1. Bounded Sequences
   (a) **Definition:** A sequence $\{a_n\}$ is bounded above if there is some M with $a_n \leq M$ for all n.
   (b) **Definition:** A sequence $\{a_n\}$ is bounded below if there is some m with $a_n \geq m$ for all n.
   (c) **Definition:** A sequence is bounded if it is both bounded above and bounded below.
   (d) **Theorem:**
   If $\{a_n\}$ converges then it is bounded.
   Intuition:
   Suppose $\{a_n\} \to a$. We'll show $\{a_n\}$ is bounded above. The idea is that eventually $\{a_n\}$ is within 1 unit of a and hence eventually less than $a+1$. Therefore if look at how high it got before that point then it never gets above that or $a+1$.
   Proof:
   Since $\{a_n\} \to a$ there exists some N so that if $n \geq N$ then $|a_n - a| < 1$, which tells us $a_n < a+1$. Let $M = \max\{a_1, a_2, ..., a_{N-1}, a+1\}$ then for all n we have $a_n \leq M$.

   $\mathcal{QED}$

   (e) **Note:** From here we can conclude that if $\{a_n\}$ is unbounded then it does not converge. Beyond that, though, nothing. For example if $\{a_n\}$ is bounded we cannot know if it converges or not.
2. Sequential Density
   (a) **Idea:** Sequential density gives us a way of obtaining a sequence converging to a real number in specific circumstances.
   (b) **Definition:** A subset $S \subseteq \mathbb{R}$ is sequentially dense if $\forall x \in \mathbb{R}$ there is a sequence $\{s_n\}$ in S converging to x.
   (c) **Theorem:**
   The subset $S \subseteq \mathbb{R}$ is sequentially dense iff it is dense.
   Proof of $\leftarrow$:
   Assume $S \subseteq \mathbb{R}$ is dense. We claim it is sequentially dense. Let $s \in \mathbb{R}$. For each $n \in \mathbb{N}$ pick $s_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ which exists because S is dense. Then since $|s_n - x| < \left|\frac{1}{n} - 0\right|$ and since $\left\{\frac{1}{n}\right\} \to 0$ we know $\{s_n\} \to x$ by the Comparison Lemma.

   $\mathcal{QED}$

   (d) **Prototypical Example:** We have $\mathbb{Q} \subseteq \mathbb{R}$ is dense and hence sequentially dense. What this says is that for any real number there's a sequence of rationals which converges to it. This is fairly clear, for example $\pi = 3.14159265...$ is irrational and the sequence $3, 3.1, 3.1, 3.14, 3.141, 3.1415, ...$ converges to π.
3. Closed Sets
   (a) **Definition:** We say $S \subseteq \mathbb{R}$ is closed if every convergent sequence in S converges to something in S.
   Note: This is just saying that a sequence in S can't converge outside of S.
   Example: The set $S = [10, 20)$ is not closed because $\left\{20 - \frac{1}{n}\right\}$ is in S but converges to 20 which is outside of S.
   Example: The set $S = [0, \infty)$ is closed. To prove this suppose $\{s_n\}$ is in S and converges to $x \notin S$ so $x < 0$. Let $\epsilon = -\frac{x}{2}$. Then there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $|s_n - x| < -\frac{x}{2}$. But then $s_n - x < -\frac{x}{2}$ and so $s_n < \frac{x}{2} < 0$, a contradiction.
   Example: The set $S = \{1\}$ (singular point) is closed because the only convergent sequence in S is the sequence $\{1\}$ which converges to $1 \in S$.
   Example: The set $S = \{1, 2\}$ (pair of points) is closed. The proof of this is a little tricker.
   (b) **Warning:** The opposite of "closed" is not "open". The word "open" means something else. The opposite of "closed" is just "not closed".