Math 410 Section 2.3: The Monotone Convergence Theorem

1. Idea: We know that if a sequence converges then it must be bounded. We also know the reverse is not true. However in the case of monotone sequences it is.

2. Definitions:
   - We say $\{a_n\}$ is monotonically (monotone) increasing if $\forall n, a_{n+1} \geq a_n$.
   - We say $\{a_n\}$ is monotonically (monotone) decreasing if $\forall n, a_{n+1} \leq a_n$.
   - A sequence is monotone if it is either.

3. **Theorem (The Monotone Convergence Theorem):**
   If $\{a_n\}$ is monotone and bounded then it converges. In addition if this is the case then:
   - If is monotone increasing then it converges to $\sup\{a_n \mid n \in \mathbb{N}\}$
   - If it is monotone decreasing then it converges to $\inf\{a_n \mid n \in \mathbb{N}\}$

   Intunition:
   For example if a sequence is monotone increasing and has an upper bound then eventually it must level off. Notice that this doesn't have to happen at the upper bound, it could happen before.
   Proof:
   Suppose $\{a_n\}$ is monotone increasing. Define S to be the set of terms in $\{a_n\}$ and define $L = \sup(S)$ which exists by the Completeness Axiom since S is bounded. We claim $\{a_n\} \to L$.
   Let $\epsilon > 0$, Since $L - \epsilon$ is not an upper bound for S (since L is the least upper bound) we know there is some N such that $a_N > L - \epsilon$ and moreover since $\{a_n\}$ is increasing for all $n \geq N$ we have $a_n > L - \epsilon$. However since L is an uppoer bound we have $a_n \leq L < L + \epsilon$ as well and hence for all $n \geq N$ we have $L - \epsilon < a_n < L + \epsilon$ and so $|a_n - L| < \epsilon$.
   The proof for monotonically decreasing is similar.

   QED

4. (a) Warning: We can't conclude the sequence converges to the bound. For example $\left\{\frac{1}{n}\right\}$ is monotone decreasing and bounded below by -17 but it certainly doesn't converge to -17.

   (b) Example: Consider the sequence defined by

   $$a_n = \sum_{k=3}^{n} \frac{1}{2^k k^2}$$

   This sequence is monotone increasing and for all n we have

   $$a_n = \sum_{k=3}^{n} \frac{1}{2^k k^2} < \sum_{k=3}^{n} \frac{1}{2^k} < \sum_{k=3}^{\infty} \frac{1}{2^k} = \frac{1}{4} \text{ (Geometric Series)}$$

   Thus it converges. As with the warning above we cannot conclude the sequence converges to $\frac{1}{4}$.

   (c) **Corollary:**
   Let $c \in \mathbb{R}$ with $|c| < 1$. Then $\{c^n\} \to 0$.
   Proof:
   Suppose $0 < c < 1$. The sequence is monotonically decreasing and bounded below by 0. It therefore converges to $L = \inf\{c^n \mid n \in \mathbb{N}\}$ so we claim $L = 0$. Suppose $L > 0$ then observe that for all n we have $c^n = \frac{c^{n+1}}{c} \geq \frac{L}{c} > L$ which contradicts L being the infimum.
   The proof for $-1 < c < 0$ is similar, the proof for $c = 0$ is trivial.

   QED