Math 410 Section 2.4: Sequential Compactness

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1. Introduction: The concept of sequential compactness helps us obtain a convergent sequence when we need one. First some preliminaries.

2. **Definition:**
   Given a sequence $\{a_n\}$ we obtain a subsequence by taking a subset of the indices and taking just those a_n. More formally we choose $n_1 < n_2 < ...$ and extract the subsequence $a_{n_1}, a_{n_2}, ...$

3. **Theorem:**
   If $\{a_n\} \to a$ then every subsequence converges to a.
   Proof:
   Omitted.

4. **Theorem:**
   Every sequence has a monotone subsequence.
   Note:
   Consider this is saying something pretty interesting, that from within any sequence at all we can extract a subsequence which is monotone.
   Proof:
   We say m is a peak index for $\{a_m\}$ if $a_n \geq a_m$ for all $n \geq m$.
   If there are infinitely many peak indices $m_1 < m_2 < ...$ then the subsequence $\{a_{m_i}\}$ is monotone decreasing.
   On the other hand if there are finitely many peak indices then choose N so that there are no peak indices beyond N. Define $n_1 = N + 1$. Since n_1 is not a peak index there is some $n_2 > n_2$ with $a_{n_2} > a_{n_1}$. However n_2 is not a peak index either so there is some $n_3 > n_2$ with $a_{n_3} > a_{n_2}$. Continuing this way we construct a monotone increasing sequence.

   $\mathcal{QED}$

5. **Theorem:**
   Every bounded sequence has a convergent subsequence.
   Proof:
   Suppose $\{a_n\}$ is bounded. By the previous theorem it has a monotone subsequence. That monotone subsequence is also bounded and hence converges by the Monotone Convergence Theorem.

   $\mathcal{QED}$

6. **Definition:**
   A set S is sequentially compact if every sequence in S has a subsequence which converges to something in S.

7. **Theorem:**
   A set S is sequentially compact iff it is closed and bounded.
   Proof:
   Suppose S is sequentially compact. We claim it is closed and bounded. To prove it is closed suppose $\{a_n\}$ is in S and converges to a. Since S is sequentially compact some subsequence of $\{a_n\}$ converges to something in S but that subsequence must converge to a so $a \in S$. To prove it is bounded suppose not, so for all n there is some $a_n > n$. Consider the sequence $\{a_n\}$. Every subsequence of $\{a_n\}$ is unbounded and hence does not converge, a contradiction.
   Suppose S is closed and bounded. We claim it is sequentially compact. Let $\{a_n\}$ be a sequence in S. We know that $\{a_n\}$is bounded and hence has a convergent subsequence which then converges to $a \in S$ because S is closed.

   $\mathcal{QED}$