Math 410 Section 3.1: Continuity

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1. **Introduction:** The idea of continuity arises from the very simple question - given a function f, as x approaches some x_0 does $f(x)$ approach $f(x_0)$?

2. **Definition:** Suppose $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Here D is the domain of the function. Let $x_0 \in D$. Then we say f is continuous at x_0 if whenever $\{x_n\}$ in D converges to x_0 we have $\{f(x_n)\}$ converging to $f(x_0)$.
   Definition: We say that f is continuous if it is continuous at each point in D.
   Note 1: What could ruin continuity at some x_0? Having just one $\{x_n\} \to x_0$ with either $\{f(x_n)\}$ converging to some other y-value or not converging at all.
   Note 2: Often continuity is defined with $\epsilon - \delta$ and we'll see this later. The benefit to using sequences is that sequences give us something concrete to use in our proofs.

3. **Examples:**

   (a) **Example:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 2x - 3$. To show that f is continuous at $x_0 = 5$ we let $\{x_n\}$ be a sequence in D with $\{x_n\} \to 5$ (note $\{x_n\}$ is unknown and arbitrary) and observe that
   $$\{f(x_n)\} = \{x_n^2 + 2x_n - 3\} \underset{*}{\to} 5^2 + 2(5) - 3 = f(5)$$
   Notice that $*$ is true by the polynomial property of the convergence of sequences, not because we're just arbitrarily plugging things in.
   Note: The same argument shows that this f is continuous at every $x_0 \in \mathbb{R}$ so we can say that f is continuous.

   (b) **Example:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
   $$f(x) = \begin{cases} 1 & \text{if } x < 3 \\ 2 & \text{if } x \geq 3 \end{cases}$$
   To show that f is not continuous at $x_0 = 3$ observe that $\left\{3 - \frac{1}{n}\right\} \to 3$ but
   $$\left\{f\left(3 - \frac{1}{n}\right)\right\} = \{1\} \to 1 \neq f(3)$$
   Notice that for all n we have $f(3 - 1/n) = 1$ because $3 - 1/n < 3$. Notice also that to ruin convergence all we needed was one sequence.
   Note: This f is continuous at every other $x_0 \in \mathbb{R}$ but we have to be delicate in doing so. For example consider $x_0 = 5$. If we take some $\{x_n\} \to 5$ and look at $\{f(x_n)\}$ we can't easily say what $f(x_n)$ equals because we don't know what x_n is for any n. It might be < 3 or ≥ 3. However we know $\{x_n\} \to 5$ so eventually it's arbitrarily close to 5. So choose $\epsilon = 1$ then there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - 5| < 1$ so that $x > 3$ so that $f(x_n) = 1$ so that for $n \geq N$ we have $\{f(x_n)\} = \{2\} \to 2 = f(5)$.

(c) **Example:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x-5} & \text{if } x \neq 5 \\ 0 & \text{if } x = 5 \end{cases}$$

To show that f is not continuous at $x_0 = 3$ observe that $\left\{5 + \frac{1}{n}\right\} \to 5$ but

$$\{f(5+1/n)\} = \left\{\frac{1}{5+1/n-5}\right\} = \{n\}$$

is unbounded and hence diverges, certainly not converging to $f(5) = 0$.
Note: This f is continuous everywhere else but we have to be declicate in our approach just like the previous question because an arbitrary sequence $\{x_n\}$ might have both $x_n = 5$ and $x_n \neq 5$ in various places.

(d) **Example:** Let $f : \mathbb{R} - \{5\} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x-5}$. Then f is continuous. This might go against intuition or might disagree with what you were taught in pre-calculus, that continuous functions must be one continuous line. This is not the case at all. If you think that this f is not continuous at $x_0 = 5$ you'd be wrong(ish). The point is that $x_0 = 5$ is not even in the domain of f so we don't have to worry about it at all!

4. **Combinations of Functions:**

(a) **Theorem:** Suppose $f, g \to D$ are continuous at $x_0 \in D$. Then $f \pm g$ and fg are continuous at x_0. In addition if $g(x) \neq 0$ for all $x \in D$ then f/g is continuous at x_0.
Proof: These follow directly from the definition of convergence of sequences.
Theorem: Suppose $f : D \to \mathbb{R}$ and $g : U \to \mathbb{R}$ where $f(D) \subseteq U$. Suppose f is continuous at $x_0 \in D$ and g is continuous at $f(x_0) \in U$. then $g \circ f : D \to \mathbb{R}$ is continuous at $x_0 \in D$.
Proof: Suppose $\{x_n\}$ is in D and converges to x_0. By continuity of f we then have $\{f(x_n)\} \to f(x_0)$. Then by continuity of g we then have $\{(g \circ f)(x_n)\} = \{g(f(x_n))\} \to g(f(x_0))$ so that $g \circ f$ is continous at x_0.