Math 410 Section 3.2: The Extreme Value Theorem

1. **Introduction:** The Extreme Value Theorem is one of the two big theorems that emerge from having continuity. The other is the Intermediate Value Theorem.
2. **Definition:** We say a function $f : D \to \mathbb{R}$ attains a maximum value provided that $f(D)$ has a maximum, meaning $\exists x_0 \in D$ such that $\forall x \in D, f(x_0) \geq f(x)$. Such an x_0 is a maximizer of f. Likewise for a minimum value.
   Note: This is not the same as $f(D)$ being bounded.
   Example: The function $f : [-1, 4] \to \mathbb{R}$ given by $f(x) = 3 - x^2$ has a maximum value of 3 with maximizer $x_0 = 0$ and has a minimum value of -13 with minimizer $x_0 = 4$.
   Example: The function $f : (0, 5] \to \mathbb{R}$ given by $f(x) = x$ has a maximum value of 5 with maximizer $x_0 = 5$ but has no minimum value. Note however that $f(D)$ is bounded below.
   Example: The function $f : [0, 5] \to \mathbb{R}$ given by

   $$f(x) = \begin{cases} \frac{1}{x(x-5)} & \text{if } x \neq 0, 5 \\ 0 & \text{if } x = 0, 5 \end{cases}$$

   has neither a maximum value nor minimum value.
3. Here is the maximum version of the Extreme Value Theorem. The minimum version can be proved either by adjusting this proof or by applying the proof to $-f(x)$.
   (a) **Lemma:** Suppose $f : D \to \mathbb{R}$ is continuous and D is closed and bounded, then $f(D)$ is bounded above.
   Proof: Suppose not. Then for all $n \in \mathbb{N}$ there is some $x_n \in D$ with $f(x_n) > n$. From here we get a sequence $\{x_n\}$ By sequential compactness choose a subsequence $\{x_{n_i}\} \to x_0 \in D$ (note $n_1 < n_2 < n_3 < ...$ are all integers). By continuity $\{f(x_{n_i})\} \to f(x_0)$ but this means that $\{f(x_{n_i})\}$ is bounded which contradicts the fact that $f(x_{n_i}) > n_i$ and $\{n_i\}$ is an increasing and unbounded sequence of integers.

   $\mathcal{QED}$

   (b) **Theorem (Extreme Value Theorem):** Suppose $f : D \to \mathbb{R}$ is continuous and D is closed and bounded. Then f attains both a maximum and minimum value.
   Proof: By the lemma, $f(D)$ is bounded above. Let $M = \sup(f(D))$. We need to find some $x_0 \in D$ with $f(x_0) = M$.
   For each $n \in \mathbb{N}$ the value $M - 1/n$ is not an upper bound for $f(D)$ and so there exists some $x_n \in D$ with $f(x_n) > M - 1/n$. In addition $f(x_n) \leq M < M + 1/n$ and so we have $M - 1/n < f(x_n) < M + 1/n$ or $|f(x_n) - M| < 1|1/n - 0|$ and so $\{f(x_n)\} \to M$ by the Comparison Lemma. By sequential compactness choose a subsequence $\{x_{n_i}\} \to x_0 \in D$. Since $\{f(x_{n_i})\}$ is a subsequence of $\{f(x_n)\}$ we also have $\{f(x_{n_i})\} \to M$ but by continuity $\{f(x_{n_i})\} \to f(x_0)$. Thus $f(x_0) = M$.

   $\mathcal{QED}$