Math 410 Section 3.3: The Intermediate Value Theorem

1. **Intermediate Value Theorem:** Suppose $f : [a, b] \to \mathbb{R}$ is continuous and c is strictly between $f(a)$ and $f(b)$ then there exists some $x_0 \in (a, b)$ such that $f(x_0) = c$.
   Proof: Note that if $f(a) = f(b)$ then there is no such c so we only need to consider $f(a) < c < f(b)$ and $f(a) > c > f(b)$. Look at the case $f(a) < c < f(b)$.
   We're going to use the Bisection Method to construct two sequences as follows:
   Define $a_1 = a$ and $b_1 = b$. Then look at $\frac{a_1+b_1}{2}$ (the midpoint) and check:
   - If $f\left(\frac{a_1+b_1}{2}\right) \leq c$ define $a_2 = \frac{a_1+b_1}{2}$ and $b_2 = b_1$.
   - If $f\left(\frac{a_1+b_1}{2}\right) > c$ define $a_2 = a_1$ and $b_2 = \frac{a_1+b_1}{2}$.

   We then repeat the procedure looking at the midpoint of $[a_2, b_2]$ and defining a_3 and b_3 accordingly and so on, to define sequences $\{a_n\}$ and $\{b_n\}$.
   Observe that $\{a_n\}$ is monotone increasing and bounded above by b and $\{b_n\}$ is monotone decreasing and bounded below by a. It follows that both converge. Moreover since

   $$\frac{b_n - a_n}{2} = \frac{1}{2^n}(b - a)$$

   we know that the difference converges to 0 and so they both converge to the same value, call it x_0. That is, $\{a_n\} \to x_0$ and $\{b_n\} \to x_0$. We then claim that $f(x_0) = c$.
   From continuity we have $\{f(a_n)\} \to f(x_0)$ and $\{f(b_n)\} \to f(x_0)$ and since for all n we have $f(a_n) \leq c$ we must have $f(x_0) \leq c$ and since for all n we have $f(b_n) > c$ we must have $f(x_0) \geq c$.
   Thus $f(x_0) = c$.
   The proof for $f(a) > c > f(b)$ is similar.

   QED

2. **Examples:**
   Example: Consider $f : [1, 5] \to \mathbb{R}$ given by $f(x) = x^2 + 4x - \frac{1}{x}$. Observe that $f(1) = 4$ and $f(5) = 44.8$. Since $c = 10$ is strictly between 4 and 44.8 we know there is some $x_0 \in (1, 5)$ with $f(x_0) = 10$.