Math 410 Section 3.4: Uniform Continuity

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1. **Intro:** The idea of uniform continuity is to present a stronger version of continuity which will be needed for some theorems. Continuity begins with a certain x_0 and asks what happens if some sequence approaches that x_0 whereas uniform continuity ask what happens if two sequences approach each other.
2. **Definition:** A function $f : D \to \mathbb{R}$ is uniformly continuous if whenever $\{u_n\}$ and $\{v_n\}$ are sequences in D with $\{u_n - v_n\} \to 0$ we must have $\{f(u_n) - f(v_n)\} \to 0$.
   Note: Uniform continuity is defined on the domain D, not at a point. It doesn't make sense to say "uniformly continuous at a point".
   Note: There is no requirement that the sequences $\{u_n\}$ and $\{v_n\}$ converge, just that they get close to one another as $n \to \infty$.
3. **Examples:**
   (a) **Example:** The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x$ is uniformly continuous. To see this suppose $\{u_n\}$ and $\{v_n\}$ are in $\mathbb{R}$ and satisfy $\{u_n - v_n\} \to 0$. Then $\{f(u_n) - f(v_n)\} = \{3u_n - 3v_n\} \to 3(0) = 0$.
   (b) **Example:** The function $f : (2, 3) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is uniformly continuous. To see this suppose $\{u_n\}$ and $\{v_n\}$ are in $\mathbb{R}$ and satisfy $\{u_n - v_n\} \to 0$. Noting that for all n we have $u_n > 2$ and $v_n > 2$ we see that

   $$|f(u_n) - f(v_n)| = \left|\frac{1}{u_n} - \frac{1}{v_n}\right| = \left|\frac{v_n - u_n}{u_n v_n}\right| < \left|\frac{v_n - u_n}{(2)(2)}\right| = \frac{1}{4}|v_n - u_n|$$

   and so $\{f(u_n) - f(v_n)\} \to 0$ by the Comparison Lemma.
   (c) **Example:** The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous. Consider the sequences $\{u_n\} = \{n + 1/n\}$ and $\{v_n\} = \{n\}$. Observe that $\{u_n - v_n\} = \{1/n\} \to 0$ but

   $$\{f(u_n) - f(v_n)\} = \{f(n + 1/n) - f(n)\} = \{n^2 + 2 + 1/n^2 - n^2\} = \{2 + 1/n^2\} \to 2 \neq 0$$

   (d) **Example:** The function $f : (0, 2) \to \mathbb{R}$ given by $f(x) = 1/x$ is not uniformly continuous. Consider the sequences $\{u_n\} = 1/n$ and $\{v_n\} = 2/n$. Observe that $\{u_n - v_n\} = \{-1/n\} \to 0$ but

   $$\{f(u_n) - f(v_n)\} = \{f(1/n) - f(2/n)\} = \{n/1 - n/2\} = \{n/2\} \not\to 0$$

4. **Theorems**
   (a) **Theorem:** If $f : D \to \mathbb{R}$ is uniformly continuous then f is continuous.
   Proof: Let $x_0 \in D$ and let $\{x_n\}$ be a sequence in D converging to x_0. Since $\{x_n\} \to x_0$ we have $\{x_n - x_0\} \to 0$ and so by uniform continuity (treating x_0 as a constant sequence $\{x_0\}$) we have $\{f(x_n) - f(x_0)\} \to 0$ and hence $\{f(x)\} \to f(x_0)$.

   $\mathcal{QED}$

   Note: The reverse is not true. A function may be continuous but not uniformly continuous. However we do get the following:

(b) **Theorem:** If $f : D \to \mathbb{R}$ is continuous and D is closed and bounded then f is uniformly continuous.

Proof: Suppose $\{u_n\}$ and $\{v_n\}$ are sequences in D satisfying $\{u_n - v_n\} \to 0$. Assume by way of contradiction that $\{f(u_n) - f(v_n)\} \not\to 0$.

This means that there is some $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there is some $n \geq N$ with $|f(u_n) - f(v_n)| \geq \epsilon$.

Define a collection of indices as follows:

- Choose $n_1 \geq N = 1$ so that $|f(u_{n_1}) - f(v_{n_1})| \geq \epsilon$.
- Choose $n_2 \geq N = n_1 + 1$ so that $|f(u_{n_2}) - f(v_{n_2})| \geq \epsilon$.
- Choose $n_3 \geq N = n_2 + 1$ so that $|f(u_{n_3}) - f(v_{n_3})| \geq \epsilon$.
- And so on.

The result is a subsequence $\{f(u_{n_k}) - f(v_{n_k})\}$ such that for all n_k we have $|f(u_{n_k}) - f(v_{n_k})| \geq \epsilon$. Next by sequential compactness we take another subsequence of $\{u_{n_k}\}$, one that converges to some $u_0 \in D$. In order to avoid notation overload we'll call this $\{u_{n_k}\}$ as well. This new subsequence satisfies $\{u_{n_k}\} \to u_0 \in [a, b]$. Then since $\{u_{n_k} - v_{n_k}\} \to 0$ we know $\{v_{n_k}\} \to u_0$ too and so by continuity $\{f(u_{n_k})\} \to f(u_0)$ and $\{f(v_{n_k})\} \to f(u_0)$ and so $\{f(u_{n_k}) - f(v_{n_k})\} \to f(u_0) - f(u_0) = 0$ which contradicts the fact that for all n we have $|f(u_{n_k}) - f(v_{n_k})| \geq \epsilon$.

$\mathcal{QED}$