Math 410 Section 3.5: The ϵ-δ Criterion for Continuity

1. **Intro:** This representation of continuity is very often used as an alternate definition. There are actually two such criteria; One matches continuity and the other matches uniform continuity.

2. (a) **Definition:** We say that $f : D \to \mathbb{R}$ satisfies the ϵ-δ criterion at $x_0 \in D$ if:
$$\forall \epsilon > 0, \exists \delta > 0 \text{ st } \forall x \in D \text{ if } |x - x_0| < \delta \text{ then } |f(x) - f(x_0)| < \epsilon$$
   (b) **Intuition:** The idea is that we can make $f(x)$ arbitrarily close to $f(x_0)$ by making x appropriately close to x_0.

   (c) **Examples:**

      i. Example: Consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x$ at $x_0 = 2$. Because the function is a line with slope 3 we can see that for the $f(x)$-values to be within ϵ of $f(2)$ we must have the x-values within $\epsilon/3$ of 2. This is borne out in the proof:

      Formal Proof: Suppose $\epsilon > 0$. Let $\delta = \epsilon/3$. Then if $|x - 2| < \epsilon/3$ then
$$|f(x) - f(2)| = |3x - 6| = 3|x - 2| < 3(\epsilon/3) = \epsilon$$
      ii. Example: Consider $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ at $x_0 = 2$. This isn't as nice as the previous problem because the slope is not constant. Instead the goal is, given $\epsilon > 0$, to choose δ so that if $|x - 2| < \delta$ then $|x^2 - 4| < \epsilon$. We can see that $|x^2 - 4| = |(x + 2)(x - 2)|$ and we can make $|x - 2|$ small but how about $|x + 2|$? Well since we can make x close to 2 we can certainly force an upper bound for $|x+2|$ by choosing an appropriate δ. For example suppose $\delta \le 1$, then if $|x - 2| < \delta \le 1$ then $-1 < x - 2 < 1$ and so $1 < x + 2 < 5$ and so $|x + 2| < 5$. If this is the case then $|(x+2)(x-2)| < 5|x-2|$ and then to make $5|x-2| < \epsilon$ we just make $|x-2| < \epsilon/5$.

      Formal Proof: Suppose $\epsilon > 0$. Let $\delta = \min\{1, \epsilon/5\}$. Then if $|x - 2| < \delta$ then
$$|x^2 - 4| = |(x + 2)(x - 2)| < 5|x - 2| < 5(\epsilon/5) = \epsilon$$
   (d) **Theorem:** Given $f : D \to \mathbb{R}$ and $x_0 \in D$, f is continuous at D iff f satisfies the ϵ-δ criterion at x_0.
   Proof of $\Longrightarrow$: Assume that f is continuous at $x_0 \in D$. We claim that f satisfies the ϵ-δ criterion at x_0. Suppose not, meaning that:
$$\exists \epsilon > 0, \forall \delta > 0, \exists x \in D \text{ with } |x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \ge \epsilon.$$
   Let $\epsilon_0 > 0$ be this ϵ. Then for each $n \in \mathbb{N}$ for $\delta = 1/n$ this give us $x_n \in D$ with $|x_n - x_0| < 1/n$ and $|f(x_n) - f(x_0)| \ge \epsilon_0 > 0$.
   We then have $\{x_n\} \to x_0$ by the Comparison Lemma and so by continuity we have $\{f(x_n)\} \to f(x_0)$ and so $\{f(x_n) - f(x_0)\} \to 0$. but this contradicts $|f(x_n) - f(x_0)| \ge \epsilon_0 > 0$ (since this is true for all x).
   Proof of $\Longleftarrow$: Assume that f satisfies the ϵ-δ criterion at x_0. We claim that f is continuous at x_0. Suppose $\{x_n\} \to x_0$. We claim $\{f(x_n)\} \to f(x_0)$. Let $\epsilon > 0$. Using the ϵ-δ criterion choose the corresponding δ. Since $\{x_n\} \to x_0$ there is some N such that if $n \ge N$ then $|x_n - x_0| < \delta$ and then $|x_n - x_0| < \delta$ implies $|f(x_n) - f(x_0)| < \epsilon$.

   QED

3. (a) **Definition:** We say that $f : D \to \mathbb{R}$ satisfies the ϵ-δ criterion on D if:
$$\forall \epsilon > 0, \exists \delta > 0 \text{ st } \forall u, v \in D \text{ if } |u - v| < \delta \text{ then } |f(u) - f(v)| < \epsilon$$
   (b) **Theorem:** Given $f : D \to \mathbb{R}$. f is uniformly continuous on D iff f satisfies the ϵ-δ criterion on D.
   Proof: Omit.