**Math 410 Section 3.6: Images and Inverses, Monotone Functions**

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1. **Intro:** Monotone function are special in several regards.
2. **Definitions**
   (a) **Definition:** The function $f : D \to \mathbb{R}$ is (monotonically, monotone) increasing if $\forall u, v \in D$ with $u < v$ we have $f(u) \leq f(v)$.
   (b) **Definition:** The function $f : D \to \mathbb{R}$ is (monotonically, monotone) decreasing if $\forall u, v \in D$ with $u < v$ we have $f(u) \geq f(v)$.
   (c) **Definition:** A function is monotone if it is either.
   (d) **Definition:** We say "strictly" if $\leq$ becomes $<$ and $\geq$ becomes $>$.
3. The first interesting thing that arises is a theorem that actually proves continuity. First just a definition to clear something up.
   (a) **Definition:** We say that $T \subseteq \mathbb{R}$ is an interval if whenever $u, v \in T$ we also have $[u, v] \in T$. For example $[2, 3)$ and $(-\infty, 10]$ are intervals but $[2, 3] \cup [4, 5]$ and $\mathbb{Q}$ and $\{2, 3\}$ are not.
   (b) **Theorem:** Suppose $f : D \to \mathbb{R}$ is monotone. If $f(D)$ is an interval then $f$ is continuous.
   **Intuition:** The idea here is that for example $f$ might be increasing and if the range is an interval then $f$ can't jump vertically at all (ruining continuity) without jumping horizontally (which doesn't ruin continuity).
   **Proof:** The proof is fairly technical and omitted for now.
   (c) **Corollary:** Suppose $I$ is an interval and $f : I \to \mathbb{R}$ is monotone. Then $f$ is continuous iff $f(I)$ is an interval.
   **Proof:** If $f(I)$ is an interval then apply the previous theorem. If $f$ is continuous then apply the IVT. This takes a few steps; try it!
4. The second thing is related to inverses of functions.
   (a) **Definition:** A function $f : D \to \mathbb{R}$ is said to be one-to-one (1-1 or injective) if $\forall y \in f(D)$ there is a unique $x \in D$ such that $f(x) = y$.
   (b) **Note:** A more standard way of understanding this is that a function is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
   (c) **Definition:** If $f : D \to \mathbb{R}$ is 1-1 then we can define the inverse function $f^{-1} : f(D) \to D$ by taking $f^{-1}(y)$ to be the unique $x$ with $f(x) = y$.
   Notice that $f^{-1}(f(x)) = x$ for all $x \in D$ and $f(f^{-1}(y)) = y$ for all $y \in f(D)$.
   (d) **Theorem:** If $f : D \to \mathbb{R}$ is strictly monotone then $f$ is 1-1 and $f^-1$ is also strictly monotone.
   **Proof:** Easy. Try it!
   (e) **Theorem:** Suppose $I$ is an interval and $f : I \to \mathbb{R}$ is strictly monotone. Then $f^{-1} : f(I) \to I$ is continuous.
   **Proof:** Since $f$ is strictly monotone so is $f^{-1}$ and so by the above Theorem since $f^{-1}(f(I)) = I$ is an interval $f^{-1}$ is continouous.

   $\mathcal{QED}$

   (f) **Use:** Notice that we know that $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is strictly monotone. Noting that $f([0, \infty)) = [0, \infty)$ It follows that the inverse function $f^{-1} : [0, \infty) \to [0, \infty)$ denoted $f^{-1}(x) = x^{1/2}$ is continuous.
   A similary argument can be made for $x^n$ for any $n \in \mathbb{N}$ and then composition can give us continuity for functions of the form $x^{n/m}$ for $n, m \in \mathbb{N}$.