Math 410 Section 3.7: Limit Points and Limits of Functions

1. **Intro:** Currently we've only encountered limits of sequences, but not of functions. These are classically mentioned in Calculus 1 so let's address them formally.
2. **Definition and Examples of Limit Points**
   (a) **Definition:** For a set $D \subseteq \mathbb{R}$ we say that $x_0 \in \mathbb{R}$ is al limit point for D if there is some sequence in $D - \{x_0\}$ which converges to x_0. Note that x_0 itself doesn't have to be in D and note that the sequence itself can't touch x_0 but must just converge to it.
   (b) **Example:** $x_0 = 2$ is a limit point for $(2, 4)$ because $\{2 + 1/n\}$ is in $(2, 4)$ and converges to 2.
   (c) **Example:** Every $x_0 \in \mathbb{R}$ is a limit point for $\mathbb{Q}$ because $\mathbb{Q}$ is sequentially dense.
3. **Definition and Examples of Limits of Functions**
   (a) **Definition:** Suppose $f : D \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ is a limit point for D. We say that
   $$\lim_{x \to x_0} f(x) = L$$
   if whenever $\{x_n\}$ is a sequence in $D - \{x_0\}$ converging to x_0 we have $\{f(x_n)\} \to L$.
   (b) **Note:** Note: x_0 has to be a limit point so at least one sequence in $D - \{x_0\}$ converges to x_0, otherwise the definition would be satisfied vacuously for every point which wasn't a limit point, for example we could define $f : [0, 2] \to \mathbb{R}$ by $f(x) = 1$ and say that $\lim_{x \to 10} f(x) = 34534$ and it would be true, vacuously, which is silly.
   (c) **Example:** Define $f : (0, 2) \to \mathbb{R}$ by $f(x) = x^2$. Then $\lim_{x \to 2} f(x) = 4$. To see this take an arbitrary $\{x_n\}$ in $(0, 2) - \{2\}$ with $\{x_n\} \to 2$ and observe that
   $$\{f(x_n)\} = \{x_n^2\} \to 2^2 = 4$$
   (d) **Example:** Define $f : \mathbb{R} \to \mathbb{R}$ by
   $$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 17 & \text{if } x = 3 \end{cases}$$
   Then $\lim_{x \to 3} f(x) = 6$. To see this take an arbitrary $\{x_n\}$ in $\mathbb{R} - \{3\}$ with $\{x_n\} \to 3$ and observe that
   $$\{f(x_n)\} = \left\{\frac{x_n^2 - 9}{x_n - 3}\right\} = \left\{\frac{(x_n - 3)(x_n + 3)}{x_n - 3}\right\} = \{x_n + 3\} \to 6$$
   Note that $\{x_n\}$ in $\mathbb{R} - \{3\}$ is required so that $x_n \neq 3$ and we can cancel $x_n - 3 \neq 0$.
4. **Combinations**
   (a) **Theorem:** Suppose $f, g : D \to \mathbb{R}$ and x_0 is a limit point of D. Suppose $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, then $\lim_{x \to x_0} (f \pm g)(x) = A \pm B$ and $\lim_{x \to x_0} (fg)(x) = AB$, and provided that $g(x) \neq 0$ for all $x \in D$ and $B \neq 0$ then $\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ also.
   (b) **Theorem:** Suppose all of the following are true:
   - $f : D \to \mathbb{R}$.
   - $g : U \to \mathbb{R}$ with $f(D) \subseteq U$.
   - x_0 is a limit point for D and $\lim_{x \to x_0} f(x) = y_0$.
   - y_0 is a limit point for U and $\lim_{y \to y_0} g(y) = L$.
   - $f(D - \{x_0\}) \subseteq U - \{y_0\}$

   Then $\lim_{x \to x_0} (g \circ f)(x) = L$.