Math 410 Section 4.1: Derivatives

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1. **Intro:** The basic premise of derivatives is to investigate the slope of a function when the function is not a line.

2. **Building the Derivative**

   (a) **Definition:** Given $x_0 \in \mathbb{R}$, a neighborhood of x_0 is an open interval (a, b) containing x_0. We may use $\pm\infty$ here.

   (b) **Secant Lines:** Suppose $x_0 \in \mathbb{R}$, suppose I is a neighborhood of x_0, and suppose $f : I \to \mathbb{R}$. For any $x \in I - \{x_0\}$ we can draw the secant line joining the points $(x_0, f(x_0))$ and $(x, f(x))$. The slope of this secant line will then equal

   $$\frac{f(x) - f(x_0)}{x - x_0}$$

   (c) **Tangent Lines:** We then investigate what happens as x gets closer to x_0. In this case the secant lines are approaching a tangent line and if the slope of the secant lines approach something this would be the slope of the tangent line.

   (d) **Definition:** Let I be a neighborhood of $x_0 \in I$. Then $f : I \to \mathbb{R}$ is said to be differentiable at x_0 if the limit

   $$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   exists. If this limit does exist then we denote if $f'(x_0)$.

   Note: The derivative is defined in terms of the limit of a function which is in turn defined as the limit of sequences. In other words to calculate $f'(x_0)$ we need to calculate $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ which means we take some arbitrary $\{x_n\}$ in $I - \{x_0\}$ with $\{x_n\} \to x_0$ and we examine the convergence of $\left\{ \frac{f(x_n)-f(x_0)}{x_n-x_0} \right\}$.

   (e) **Definition:** If f is differentiable at every point in I then we simply say f is differentiable on I. In this case we would have some $f'(x_0)$ for each $x_0 \in I$ and we usually write $f'(x)$ instead.

3. **Examples**

   (a) Example: We show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 7$ is differentiable at $x_0 = 3$. We investigate the limit

   $$\lim_{x \to 3} \frac{(2x + 7) - (2(3) + 7)}{x - 3}$$

   To do this we take a sequence $\{x_n\}$ in $\mathbb{R} - \{3\}$ with $\{x_n\} \to 3$ and observe that

   $$\left\{ \frac{(2x_n + 7) - (2(3) + 7)}{x_n - 3} \right\} = \left\{ \frac{2(x_n - 3)}{x_n - 3} \right\} = \{2\} \to 2$$

   and so $f'(3) = 2$.

   (b) Example: We show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is differentiable at $x_0 = 7$. We investigate the limit

   $$\lim_{x \to 7} \frac{x^2 - 7^2}{x - 7}$$

   . To do this we take a sequence $\{x_n\}$ in $\mathbb{R} - \{7\}$ with $\{x_n\} \to 7$ and observe that

   $$\left\{ \frac{x_n^2 - 7^2}{x_n - 7} \right\} = \{x_n + 7\} \to 2x_0$$

   and so $f'(7) = 2(7)$.

(c) Example: We show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is not differentiable at $x_0 = 0$. Observe that $\{1/n\} \to 0$ and

$$\left\{ \frac{f(1/n) - f(0)}{1/n - 0} \right\} = \left\{ \frac{|1/n|}{1/n} \right\} = \{1\} \to 1$$

but observe that $\{-1/n\} \to 0$ and

$$\left\{ \frac{f(-1/n) - f(0)}{-1/n - 0} \right\} = \left\{ \frac{|-1/n|}{-1/n} \right\} = \{-1\} \to -1$$

4. **Theorems:**

(a) **Theorem:** A similar proof to the above will show that if $f(x) = mx + b$ then $f'(x_0) = m$ for all x_0 or more commonly $f'(x) = m$.

(b) **Theorem:** A similar proof to the above will show that if $f(x) = x^2$ then $f'(x_0) = 2x_0$ for all x_0 or more commonly $f'(x) = 2x$.

(c) **Theorem:** If $n \in \mathbb{Z}$ with $n \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

(d) **Theorem:** If I is a neighborhood of x_0 and $f : I \to \mathbb{R}$ is differentiable at x_0 then f is continuous at x_0.
Proof: We need to prove that for any $\{x_n\} \to x_0$ that $\{f(x_n)\} \to f(x_0)$. To that end suppose $\{x_n\} \to x_0$. Now observe that for any n:

- If $x_n = x_0$ then $f(x_n) - f(x_0) = 0$
- f $x_n \neq x_0$ then $f(x_n) - f(x_0) = (x_n - x_0)\left(\frac{f(x_n)-f(x_0)}{x_n-x_0}\right)$

In the first case the terms are all 0 and in the second case the terms converge to $0f'(x_n) = 0$. It follows that $\{f(x_n) - f(x_0)\} \to 0$ and so $\{f(x_n)\} \to f(x_0)$.

(e) **Theorem:** Let I be a neighborhood of x_0 and suppose that $f, g : I \to \mathbb{R}$ are differentiable at x_0. Then:

i. $f \pm g$ is differentiable at x_0 and $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

ii. fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
Proof: We claim that:

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

Suppose $\{x_n\}$ is in $I - \{x_0\}$ and $\{x_n\} \to x_0$. Observe that:

$$\begin{aligned}
\left\{ \frac{(fg)(x_n) - (fg)(x_0)}{x_n - x_0} \right\} &= \left\{ \frac{f(x_n)g(x_n) - f(x_0)g(x_0)}{x_n - x_0} \right\} \\
&= \left\{ \frac{f(x_n)g(x_n) - f(x_0)g(x_n) + f(x_0)g(x_n) - f(x_0)g(x_0)}{x_n - x_0} \right\} \\
&= \left\{ \frac{f(x_n) - f(x_0)}{x_n - x_0} g(x_n) + f(x_0) \frac{g(x_n) - g(x_0)}{x_n - x_0} \right\}
\end{aligned}$$

and then observe that as $\{x_n\} \to x_0$ we have:

$$\left\{ \frac{f(x_n) - f(x_0)}{x_n - x_0} g(x_n) + f(x_0) \frac{g(x_n) - g(x_0)}{x_n - x_0} \right\} \to f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

iii. $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g'(x_0)^2}$$

(f) **Theorem:** If $n \in \mathbb{Z}$ with $n < 0$ and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^n$ then $f'(x) = nx^{n-1}$.