Math 410 Section 4.2: Derivatives of Inverses and Compositions

1. **Derivatives of Inverses**

   (a) **Theorem:** Let I be a neighborhood of x_0 and let $f : I \to \mathbb{R}$ be strictly monotone and continuous. Suppose f is differentiable at x_0 and that $f'(x_0) \neq 0$. Define $y_0 = f(x_0)$ then $f^{-1} : f(I) \to \mathbb{R}$ is differentiable at y_0 and

   $$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

   Proof: By the IVT we know $f(I)$ is an interval hence a neighborhood of y_0. Let $\{y_n\}$ be a sequence in $f(I) - \{y_0\}$ converging to y_0. Thus for each n we have $y_n = f(x_n)$ for some $x_n \in I - \{x_0\}$ (missing x_0 because f is strictly monotone).
   Observe that $\{y_n\} \to y_0$ and so by continuity of f^{-1} (previous theorem - a strictly monotone function on an interval has a continuous inverse) we know that $\{f^{-1}(y_n)\} \to f^{-1}(y_0)$ which is $\{x_n\} \to x_0$ and so

   $$\left\{\frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}\right\} = \left\{\frac{x_n - x_0}{f(x_n) - f(x_0)}\right\} \to \frac{1}{f'(x_0)}$$

   (b) **Theorem:** For $n \in \mathbb{N}$ define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = x^{1/n}$. Then f is continuous and

   $$f'(x) = \frac{1}{n}x^{1/n-1}$$

   Proof: Define $g(x) = x^n$ and apply the above theorem to g noting that $f = g^{-1}$.

2. **Derivatives of Compositions**

(a) **Theorem (The Chain Rule):** Let I be a neighborhood of x_0 and suppose $f : I \to \mathbb{R}$ is differentiable at x_0. Let J be an open neighborhood such that $f(I) \subseteq J$ and suppose that $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$. Then $g \circ f : I \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))g'(x_0)$$

Proof: Let $\{x_n\}$ be a sequence in $I - \{x_0\}$ with $\{x_n\} \to x_0$. Let's examine:

$$\left\{ \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \right\}$$

Observe that we have the following equality $A = BC$ where expression B is defined in cases according to n:

$$\underbrace{\frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0}}_{A} = \underbrace{\left[\begin{array}{ll} \frac{g(f(x_n))-g(f(x_0))}{f(x_n)-f(x_0)} & \text{if } f(x_n) - f(x_0) \neq 0 \\ g'(f(x_0)) & \text{if } f(x_n) - f(x_0) = 0 \end{array} \right]}_{B} \underbrace{\left[\frac{f(x_n) - f(x_0)}{x_n - x_0} \right]}_{C}$$

The reason these are equal is:

- If $f(x_n) - f(x_0) \neq 0$ then the denominator of B and numerator of C cancel, yielding $A = BC$.
- If $f(x_n) - f(x_0) = 0$ then B equals $g'(f(x_0))$ and the numerator of $C = 0$ so $BC = 0$ but also $g(f(x_n)) - g(f(x_0)) = 0$ so $A = 0$ and so $A = BC$.

Now check out the limits of each of B and C:

- For C we have $\lim\limits_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n-x_0} = f'(x_0)$ since f is differentiable at x_0.
- For B notice that terms with $f(x_n) - f(x_0) = 0$ equal $g'(f(x_0))$ and terms with $f(x_n) - f(x_0) \neq 0$ approach $g'(f(x_0))$ because $\{x_n\} \to x_0$ along with f being continuous (since it's differentiable) implies $\{f(x_n)\} \to f(x_0)$. All together we see that the limit is $g'(f(x_0))$.

Thus all together the limit of the right side is $g'(f(x_0))f'(x_0)$ and so:

$$\left\{ \frac{g(f(x_n)) - g(f(x_0))}{x_n - x_0} \right\} \to g'(f(x_0))f'(x_0)$$

which is what we wished to prove.