Math 410 Section 4.3: The Mean Value Theorem and Consequences

1. **The Mean Value Theorem**

   (a) **Introduction:** The MVT is far more important than you may suspect. It provided a fundamental connection between a function and its derivative and it underlies many properties of the integral. However first we needs some preliminaries.

   (b) **Lemma:** Let I be a neighborhood of x_0 and suppose $f : I \to \mathbb{R}$ is differentiable at x_0. If x_0 is either a maximizer or minimizer for f then $f'(x_0) = 0$

   Proof Suppose f is a maximizer. Since f is differentiable at x_0 and since I is a neighborhood choose a sequence $\{x + 1/n\}$ for n starting large enough to be in I. we know that:

   $$\left\{ \frac{f(x_0 + 1/n) - f(x_0)}{x_0 + 1/n - x_0} \right\} \to f'(x_0)$$

   Since x_0 is a maximizer the numerator is nonpositive and so the terms of the sequence are in $(-\infty, 0]$ which is closed we must have $f'(x_0) \leq 0$. A similar argument with $\{x - 1/n\}$ shows that $f'(x_0) \geq 0$ and so $f'(x_0) = 0$. The proof for a minimizer is similar.

   (c) **Rolle's Theorem:** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover assume $f(a) = f(b)$. Then there is a point $x_0 \in (a, b)$ at which $f'(x_0) = 0$.

   Proof: By the EVT we know f assumes both a maximum and a minimum. If both occur at endpoints then the function is constant and so $f'(x) = 0$ everywhere. Otherwise let x_0 be either a maximizer or minimizer not at an endpoint then since (a, b) is a neighborhood of x_0 the lemma tells us that $f'(x_0) = 0$.

   (d) **The Mean Value Theorem:** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $x_0 \in (a, b)$ at which

   $$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

   Proof: Define $h(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right] x$. Observe that

   $$h(a) = f(a) - \left[\frac{f(b) - f(a)}{b - a}\right] a = \frac{f(a)(b - a) - af(b) + af(a)}{b - a} = \frac{bf(a) - af(b)}{b - a}$$

   and

   $$h(b) = f(b) - \left[\frac{f(b) - f(a)}{b - a}\right] b = \frac{f(b)(b - a) - bf(b) + bf(a)}{b - a} = \frac{bf(a) - af(b)}{b - a}$$

   So we can apply Rolle's Theorem to obtain a point $x_0 \in (a, b)$ with $h'(x_0) = 0$. However then

   $$0 = h'(x_0) = f'(x_0) - \left[\frac{f(b) - f(a)}{b - a}\right]$$

   and we're done.

2. **Consequences**

(a) **The Identity Criterion Lemma:** Let I be an open interval and suppose $f : I \to \mathbb{R}$ is differentiable. Then f is constant iff $f'(x) = 0$ for all $x \in I$.
Proof: If f is constant then the definition of the derivative shows that $f'(x) = 0$ for all $x \in I$.
Suppose that $f'(x) = 0$ for all $x \in I$. If f is not constant then there are $a < b$ in I with $f(a) \neq f(b)$. Since f is differentiable on I it is continuous on $[a, b]$ and differentiable on (a, b) and hence by the MVT there is some $x_0 \in (a, b)$ with

$$f(x_0) = \frac{f(b) - f(a)}{b - a} \neq 0$$

which is a contradiction. Therefore f is constant.

(b) **The Identity Criterion:** Let I be an open interval and suppose $f, g : I \to \mathbb{R}$ are differentiable. Then f and g differ by a constant iff $f'(x) = g'(x)$ for all $x \in I$.
Proof: Define $g(x) = f(x) - g(x)$ and apply the ICL.

(c) **Strict Monotonicity Criterion:** Let I be an open interval and suppose $f : I \to \mathbb{R}$ is differentiable. Suppose $f'(x) > 0$ for all $x \in I$. Then f is strictly increasing. Likewise suppose $f'(x) < 0$ for all $x \in I$. Then f is strictly decreasing.
Proof: For the $f'(x) > 0$ case suppose $a < b$ are in I. Since f is differentiable on I it is continuous on $[a, b]$ and differentiable on (a, b) and hence by the MVT there is some $x_0 \in (a, b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(x_0) > 0$$

and the result follows. The $f'(x) < 0$ case is similar.