Math 410 Section 4.4: The Cauchy Mean Value Theorem and Consequences

1. **Introduction:** The Cauchy MVT is a slight adaptation of the MVT which we'll use once or twice later on. It's not nearly as intuitive as the MVT however.

2. **The Cauchy Mean Value Theorem:** Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Moreover assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there is some $x_0 \in (a, b)$ such that:
$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$
Note: The intuition here is that really we have:
$$\frac{f'(x_0)}{g'(x_0)} = \frac{(f(b) - f(a))/(b - a)}{(g(b) - g(a))/(b - a)}$$
so the CMVT is saying that the ratio of instantaneous slopes equals the ratio of the slopes joining the ending lines.
Proof: Define
$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g(x)$$
Noting that $h(a) = h(b)$ (some algebra) we apply Rolle's Theorem to obtain $x_0 \in (a, b)$ with:
$$\begin{aligned} h'(x_0) &= 0 \\ f'(x_0) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g'(x_0) &= 0 \\ \frac{f'(x_0)}{g'(x_0)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned}$$

3. **Function Bounding Theorem:** Let I be an open interval and $n \in \mathbb{N}$ and suppose that $f : I \to \mathbb{R}$ has n derivatives. Suppose also that:
$$f(x_0) = f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0$$
Then for each $x \neq x_0$ in I there is some z strictly between x_0 and x with:
$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$
Note: Loosely speaking this theorem is stating that a function is controlled by its first nonzero derivative. We'll see a specific example of this in action soon.
Proof: Define $g(x) = (x - x_0)^n$ for all $x \in I$. A quick calculation shows that
$$g(x_0) = g'(x_0) = ... = g^{(n-1)}(x_0) = 0 \text{ and } g^{(n)}(x_0) = n!$$
Without loss of (much) generality assume $x_0 < x$. Observe that:
$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(x_1)}{g'(x_1)} = \frac{f'(x_1) - f'(x_0)}{g'(x_1) - g'(x_0)} = \frac{f''(x_2)}{g'(x_2)} = \frac{f''(x_2) - f''(x_0)}{g''(x_2) - g''(x_0)}$$
$$= ... = \frac{f^{(n-1)}(x_{n-1})}{g^{(n-1)}(x_{n-1})} = \frac{f^{(n-1)}(x_{n-1}) - f^{(n-1)}(x_0)}{g^{(n-1)}(x_{n-1}) - g^{(n-1)}(x_0)} = \frac{f^{(n)}(x_n)}{g^{(n)}(x_n)} = \frac{f^{(n)}(x_n)}{n!}$$
Where $x_1 \in (x_0, x)$ is chosen by the CMVT applied to f and g, $x_2 \in (x_1, x)$ is chosen by the CVMT applied to f' and g', and so on. Letting $z = x_n$ we have our claim.

4. **Examples**

Here is an example which illustrates what the theorem tells us, followed by another which has a slight tweak.

(a) **Example:** Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $f(1) = 0$, $f'(1) = 0$ and $f''(x) \leq 3$ for all x. How big could $f(5)$ possibly be?

Solution: Letting $x_0 = 1$, $x = 5$ and $n = 2$ the FBT states that there is some $z \in (1, 5)$ such that

$$f(3) = \frac{f''(z)}{2!}(5-1)^2 = 8f''(z)$$

Since $f''(z) \leq 3$ we then have $f(3) = 8f''(z) \leq 24$.

(b) **Example:** Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $f(2) = 3$, $f'(2) = 0$, $f''(2) = 0$ and $0.1 \leq f'''(x) \leq 0.12$ for all x. What are the restrictions on $f(2.7)$?

Solution: Letting $x_0 = 2$, $x = 2.7$ and $n = 3$ we notice we can't apply the FBT immediately because $f(x_0) \neq 0$. Define $g(x) = f(x) - 3$ and notice that $g(2) = 0$, $g'(2) = 0$, $g''(2) = 0$ and $0.1 \leq g'''(x) \leq 0.12$ for all x so we apply the FBT to $g(x)$ instead to get some $z \in (2, 2.7)$ such that

$$g(2.7) = \frac{g'''(z)}{3!}(2.7-2)^3 = \frac{343}{6000}g'''(z)$$

Since $0.1 \leq g'''(z) \leq 0.12$ we then get

$$(343/6000)(0.1) \leq g(2.7) \leq (343/6000)(0.12)$$

so that

$$(343/6000)(0.1) \leq f(2.7) - 3 \leq (343/6000)(0.12)$$

and so

$$3 + (343/6000)(0.1) \leq f(2.7) \leq 3 + (343/6000)(0.12)$$