Math 410 Section 6.1: Darboux Sums - Lower and Upper Integrals

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1. **Introduction:** The overall goal of the chapter is to establish the two Fundamental Theorems of Calculus. The first states that antiderivatives can be used to evaluate integrals and the second states that integrals can be used to construct antiderivatives.
   It may take a minute or two to remember that antiderivatives and integrals are not the same thing at all and it's the FTOC which connects them.

2. **Upper and Lower Darboux Sums**

   (a) **Partitions:** Given a closed interval $[a, b]$, a partition of $[a, b]$ is a division into subintervals. More specifically it is a choice of $a = x_0 < x_1 < x_2 < ... < x_n = b$. We usually write a partition as $P = \{x_0, x_1, ..., x_n\}$.

   (b) **Darboux Sums:** Suppose $f : [a, b] \to \mathbb{R}$ is bounded and P is a partition of $[a, b]$. Then the Lower Darboux Sum is:

   $$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \text{ where } m_i = \inf\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$$

   and the Upper Darboux Sum is:

   $$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \text{ where } M_i = \sup\{f(x) \,|\, x \in [x_{i-1}, x_i]\}$$

   (c) **Note:** These are remeniscent of the lower and upper sums encountered in most calculus courses but the use of inf and sup makes them more flexible.

   (d) **Example:** A good visual one suffices, especially one in which one of the subintervals has a hole in the function so the inf (or sup) exists but the min (or max) does not. Otherwise it's just like lower and upper sums.

   (e) **Theorem:** Suppose $f : [a, b] \to \mathbb{R}$ is bounded with lower and upper bounds m and M respectively. Then

   $$m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$$

   Proof: Since for each subinterval we have $m \le m_i \le M_i \le M$ we then get the inequalities:

   $$m(x_i - x_{i-1}) \le m_i(x_i - x_{i-1}) \le M_i(x_i - x_{i-1}) \le M(x_i - x_{i-1})$$
   $$\sum_{i=1}^{n} m(x_i - x_{i-1}) \le \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} M(x_i - x_{i-1})$$
   $$m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$$

   (f) **Definition:** If P is a partition of $[a, b]$ then a refinement of P is a partition containing at least the x-values in P and maybe more. In other words we just add more cuts.

   (g) **Theorem (The Refinement Theorem):** Suppose $f : [a, b] \to \mathbb{R}$ is bounded, P is a partition of $[a, b]$ and P^* is a refinement of P. Then

   $$L(f, P^*) \ge L(f, P)$$

   and

   $$U(f, P^*) \le U(f, P)$$

   Intuition: The intuition here is that as we refine the partition lower sums go up and upper sums go down.
   Proof: Omitted.

(h) **Theorem:** Suppose $f : [a,b] \to \mathbb{R}$ is bounded and P_1 and P_2 are both partitions of $[a,b]$. Then

$$L(f,P_1) \le U(f,P_2)$$

Proof: Let P^* be the partition obtained by using all the x-values in both P_1 and P_2 combined. Then

$$L(f,P_1) \le L(f,P^*) \le U(f,P^*) \le U(f,P_2)$$

3. **Upper and Lower Integrals**

(a) **Upper and Lower Integrals:** Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then we define the lower integral of f on $[a,b]$ as:

$$\underline{\int_a^b} f = \sup(L) = \mathrm{lub}(L) \text{ where } L = \{L(f,P) \mid P \text{ is a partition of } [a,b]\}$$

and the upper integral of f on $[a,b]$ as:

$$\overline{\int_a^b} f = \inf(U) = \mathrm{glb}(U) \text{ where } U = \{U(f,P) \mid P \text{ is a partition of } [a,b]\}$$

(b) **Theorem:** Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then

$$\underline{\int_a^b} f \le \overline{\int_a^b} f$$

Proof: For any partition P of $[a,b]$ we know $U(f,P)$ is greater than or equal to every lower sum so $U(f,P)$ is an upper bound for L and since $\underline{\int_a^b} f = \mathrm{lub}(L)$ we must have $\underline{\int_a^b} f \le U(f,Q)$. However since this is true for all P we know that $\underline{\int_a^b} f$ is a lower bound for U and since $\overline{\int_a^b} f = \mathrm{glb}(U)$ we must have

$$\underline{\int_a^b} f \le \overline{\int_a^b} f$$

(c) **Note:** Calculating lower and upper Darboux integrals using partitions is extremely difficult because it requires and understanding of what the lower and upper Darboux sums would be for every partition in order to make sense of the required sup and inf.

(d) **Example:** Define $f : [0,2] \to \mathbb{R}$ by $f(x) = 3$. Then $\underline{\int_0^2} f = 6$ and $\overline{\int_0^2} f = 6$.

(e) **Example:** Define $f : [0,1] \to \mathbb{R}$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Then $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$.