Math 410 Section 6.3: Additivity, Monotonicity, Linearity

1. **Introduction:** Just like with differentiability it's helpful to establish rules for when functions are and are not integrable.

2. **Lemma:** Suppose f is integrable and $\{P_n\}$ is an Archimedean sequence of partitions. Then if P_n^* is a refinement of P_n for each n then $\{P_n^*\}$ is also an Archimedean sequence of partitions.
   Proof: We know that:
   $$\{U(f,P_n) - L(f,P_n)\} \to 0$$
   and by the Refinement Theorem:
   $$U(f,P_n^*) - L(f,P_n^*) \leq U(f,P_n) - L(f,P_n)$$
   Then by the Comparison Lemma we have
   $$\{U(f,P_n^*) - L(f,P_n^*)\} \to 0$$

3. **Theorem (Additivity Over Intervals):** Suppose f is integrable on $[a,b]$ and let $c \in (a,b)$. Then f is integrable on $[a,c]$ and on $[c,b]$ and
   $$\int_a^b f = \int_a^c f + \int_c^b f$$
   Proof: Let $\{P_n\}$ be an Archimedean sequence of partitions for f. By the above lemma adding $x = c$ to each P_n still results in an Archimedean sequence of partitions so we just assume that $x = c$ is in each P_n. Write $P_n = P_n' \cup P_n''$ where P_n' and P_n'' are the partitions induced by P_n on just $[a,c]$ and $[c,b]$ respectively. By the definition of upper and lower sums we have:
   $$U(f,P_n) - L(f,P_n) = [U(f,P_n') - L(f,P_n')] + [U(f,P_n'') - L(f,P_n'')]$$
   Since the second bracket on the right is nonnegative we have
   $$U(f,P_n) - L(f,P_n) \geq U(f,P_n') - L(f,P_n')$$
   So that by the Comparison Lemma $\{P_n'\}$ is an Archimedean sequence of partitions for f on $[a,c]$ so f is integrable on $[a,c]$ and $\{U(f,P')\} \to \int_a^c f$. A similar argument shows that $\{P_n''\}$ is an Archimedean sequence of partitions for f on $[c,b]$ so f is integrable on $[c,b]$ and $\{U(f,P'')\} \to \int_c^b f$. Therefore since
   $$\{U(f,P_n)\} \to \int_a^b f$$
   and
   $$\{U(f,P_n)\} = \{U(f,P_n') + U(f,P_n'')\} \to \int_a^c f + \int_c^b f$$
   we have the result.

4. **Theorem (Monotonicity):** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable and for all $x \in [a, b]$ we have $f(x) \leq g(x)$. Then

$$\int_a^b f \leq \int_a^b g$$

Proof: Take an Archimedean sequence of partitions for f and one for g. For each n take the union P_n of the corresponding partitions. By the above lemma the resulting $\{P_n\}$ is an Archimedean sequence of partitions for both f and g. From here we get:

$$\{U(g, P_n) - U(f, P_n)\} \to \int_a^b g - \int_a^b f$$

However since $f(x) \leq g(x)$ we have $U(g, P_n) - U(f, P_n) \geq 0$ and therefore since $[0, \infty)$ is closed we have

$$\int_a^b g - \int_a^b f \geq 0$$

5. **Theorem (Linearity):** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable and $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha f + \beta g$ is integrable on $[a, b]$ and

$$\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$$

Proof: Omit (several pages).