Math 410 Section 6.4: Continuity and Integrability

1. **Theorem:** Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f is integrable on $[a, b]$.

   Lemma: Suppose $f : [a, b] \to \mathbb{R}$ is continuous and suppose $P = \{x_0, ..., x_n\}$ is a partition. Then there is some partition interval $[x_{i-1}, x_i]$ containing two points $u, v \in [x_{i-1}, x_i]$ such that

   $$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a]$$

   Proof of Lemma: Choose i so that $M_i - m_i$ is maximum. For this particular subinterval since f is continuous this implies that this M_i and m_i are actuallya maximum and a minimum obtainable by the function so choose u, v in that subinterval with $f(u)$ equal to the minimum and $f(v)$ equal to the maximum. Then

   $$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_{i-1} - x_i) \\ &\leq \sum_{i=1}^{n} (f(v) - f(u))(x_{i-1} - x_i) \\ &= (f(v) - f(u)) \sum_{i=1}^{n} (x_{i-1} - x_i) \\ &= (f(v) - f(u))(b - a) \end{aligned}$$

   Proof of Theorem: Let $\{P_n\}$ be the regular partition. For each n use the lemma to choose u_n and v_n such that

   $$0 \leq U(f, P_n) - L(f, P_n) \leq [f(v_n) - f(u_n)][b - a]$$

   As $n \to \infty$ the width of each subinterval goes to 0 which implies that $\{v_n - u_n\} \to 0$. Since f is continuous on a closed and bounded interval it is uniformly continuous and hence $\{f(v_n) - f(u_n)\} \to 0$. It follows from the Comparison Lemma that

   $$\{U(f, P_n) - L(f, P_n)\} \to 0$$

   and so f is integrable on $[a, b]$.

Theorem: Suppose that $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and continuous on (a, b). Then f is integrable on $[a, b]$ and the value of the integral does not depend on $f(a)$ and $f(b)$.
Proof: First choose N so that $a < a + \frac{1}{N} < b - \frac{1}{N} < b$. For what follows assume that $n \geq N$ and so $a < a + \frac{1}{n} < b - \frac{1}{n} < b$ too. Next, since f is integrable on $\left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ each of these intervals has an Archimedean sequence of partitions and so from each sequence choose some P'_n satisfying

$$U(f, P'_n) - L(f, P'_n) < \frac{1}{n}$$

Taking all these n together by the Comparison Lemma note that:

$$\{U(f, P'_n) - L(f, P'_n)\} \to 0$$

For each n add $x = a$ and $x = b$ to P'_n to get a sequence of partitions $\{P_n\}$ of $[a, b]$. In other words:

$$P_n = \left\{x_0 = a, x_1 = a + \frac{1}{n}, x_2, ..., x_{n-1} = b - \frac{1}{n}, x_n = b\right\}$$

Observe that if M and m are the maximum and minimum of f on $[a, b]$ then:

$$U(f, P_n) \leq M\left(\frac{1}{n}\right) + U(f, P'_n) + M\left(\frac{1}{n}\right)$$

and

$$L(f, P_n) \leq m\left(\frac{1}{n}\right) + L(f, P'_n) + m\left(\frac{1}{n}\right)$$

and so

$$U(f, P_n) - L(f, P_n) \leq \frac{2M - 2m}{n} + [U(f, P'_n) - L(f, P'_n)]$$

From here we have

$$\{U(f, P_n) - L(f, P_n)\} \to 0$$

by the Comparison Lemma and so f is integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} U(f, P_n)$.

Next, noting that

$$U(f, P_n) - U(f, P'_n) \leq \frac{2M}{n}$$

it follows that

$$\{U(f, P_n) - U(f, P'_n)\} \to 0$$

and so

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} U(f, P'_n)$$

and since none of the $U(f, P'_n)$ depend on $f(a)$ or $f(b)$, neither does the right-hand limit and hence neither does $\int_a^b f$.