Math 410 Section 6.5: The First Fundamental Theorem of Calculus

1. **Introduction:** The First Fundamental Theorem of Calculus deals with integrating derivatives. More intuitively it states that the antiderivative of a function may be used to calculate the integral. It's this version that's used most frequently in standard calculus.

2. **The First Fundamental Theorem of Calculus:** Suppose $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose that $F' : (a, b) \to \mathbb{R}$ is both continuous and bounded. Then:

$$\int_a^b F' = F\Big|_a^b = F(b) - F(a)$$

Example: The function $F(x) = x^2$ with $F'(x) = 2x$ satisfies these hypotheses on the interval $[1, 5]$. Consequently:

$$\int_1^5 2x = x^2\Big|_1^5 = 25 - 1 = 24$$

Proof: By the second theorem in the previous section we can define $F'(a)$ and $f'(b)$ however we like and the new function F' is integrable on $[a, b]$ and the value of the integral $\int_a^b F'$ does not depend on these values.

For a partition P and for a subinterval $[x_{i-1}, x_i]$ the function $F : [x_{i-1}, x_i] \to \mathbb{R}$ is continous on $[x_{i-1}, x_i]$ and differentiable on (x_{i-1}, x_i) and hence by the MVT there is some $c_i \in (x_{i-1}, x_i)$ satisfying

$$F'(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$$

If we let m_i and M_i be the inf and sup of F' on each subinterval then we then have

$$m_i(x_i - x_{i-1}) \leq F(x_i) - F(x_{i-1}) \leq M_i(x_i - x_{i-1})$$

and if we sum over all subintervals we have

$$L(F', P) \leq F(b) - F(a) \leq U(F', P)$$

where the middle term collapses as a telescoping sequence.

Since this is true for any partition P we can see that $F(b) - F(a)$ is a lower bound on the set of upper sums and also an upper bound on the set of lower sums and so

$$\int_a^b F' = \overline{\int_a^b} F' = \text{glb}(U) \geq F(b) - F(a)$$

and

$$\int_a^b F' = \underline{\int_a^b} F' = \text{lub}(L) \leq F(b) - F(a)$$

so that

$$\int_a^b F' = F(b) - F(a)$$

$\mathcal{QED}$

3. **Note 1:** Realize that in order to use this to evaluate some arbitrary $\int_a^b f$ that f must be the derivative of some function. More specifically f must be continuous and bounded and there must be some $F : [a, b] \to \mathbb{R}$ which is continuous on $[a, b]$ and differentiable on (a, b) with $F' = f$ on (a, b).

   It's entirely possible for these criteria not to be met and for $\int_a^b f$ to still exist. For example consider the function

$$f(x) = \begin{cases} 0 & \text{for } x \in [-1, 0) \\ 1 & \text{for } x \in [0, 1] \end{cases}$$

   This is a step function which is hence integrable and in fact it's not hard to see that

$$\int_{-1}^{1} f = 1$$

   However f has no antiderivative on $(-1, 1)$. More specifically there is no $F : [-1, 1] \to \mathbb{R}$ with $F' = f$ on $(-1, 1)$. To see this observe that if there were such an F then since $F'(x) = f(x) = 0$ on $(-1, 0)$ we must have $F(x) = C$ on $(-1, 0)$ by the Identity Criterion.

   At this point F being differentiable at $x = 0$ would then require the derivative to exist for every sequence so examining the sequence $\{-1/n\}$ we would then have:

$$1 = f(0) = F'(0) = \lim_{n\to\infty} \frac{F(-1/n) - F(0)}{-1/n - 0} = \lim_{n\to\infty} \frac{C - F(0)}{-1/n - 0} = \lim_{n\to\infty} n(F(0) - C)$$

   However the only way that this limit can exist is if $F(0) = C$ in which case the limit is 0.

4. **Note 2:** Even if f is the antiderivative of a function this doesn't mean that the antiderivative can be found in any useful manner. For example consider the integral

$$\int_0^1 \frac{1}{1 + x^4}$$

   The function $f(x) = \frac{1}{1+x^4}$ is continuous on $[0, 1]$ and hence integrable so the integral exists. Moreover this function does have an antiderivative, meaning there is some $F(x)$ defined on $[0, 1]$ with $F'(x) = \frac{1}{1+x^4}$ on $(0, 1)$. However this F has no particularly nice closed form, meaning we cannot write it down in simple terms and use it to evaluate the integral.