Math 410 Section 6.6: The Second Fundamental Theorem of Calculus

1. **Introduction:** The Second Fundamental Theorem of Calculus deals with constructing antiderivatives. More intuitively it states that under certain circumstances an integral may be used to construct an antiderivative of a function.

2. **The Mean Value Theorem for Integrals:** Suppose $f : [a, b] \to \mathbb{R}$ is continuous, then $\exists x_0 \in [a, b]$ such that

$$\int_a^b f = f(x_0)(b - a)$$

Proof: Since f is continous apply the EVT to choose x_m and x_M in $[a, b]$ satisfying

$$f(x_m) \leq f(x) \leq f(x_M)$$

for all $x \in [a, b]$. We then have

$$f(x_m)(b - a) \leq \int_a^b f \leq f(x_M)(b - a)$$

and so

$$f(x_m) \leq \frac{1}{b-a} \int_a^b \leq f(x_M)$$

and now apply the IVT to get x_0 between x_m and x_M with

$$f(x_0) = \frac{1}{b-a} \int_a^b$$

3. **Preliminary Theorem**

(a) **Theorem (Continuity of the Integral):** Suppose the function $f : [a, b] \to \mathbb{R}$ is integrable. For all $x \in [a, b]$ define the function

$$F(x) = \int_a^x f$$

Then F is continuous.
Proof: Since f is integrable it is bounded so there is some $M > 0$ with $|f(x)| \leq M$ for all $x \in [a, b]$.
Let $x_0 \in [a, b]$ and suppose $\{x_n\} \to x_0$. We claim that $\{F(x_n)\} \to F(x_0)$.
Observe that if $x_0 < x_n$ then

$$F(x_n) - F(x_0) = \int_a^{x_n} f - \int_a^{x_0} f = \int_{x_0}^{x_n} f$$

at which point we then have

$$|F(x_n) - F(x_0)| = \left| \int_{x_0}^{x_n} f \right| \leq M\,|x_n - x_0|$$

and if $x_n < x_0$ then

$$F(x_n) - F(x_0) = -\left[F(x_0) - F(x_n)\right] = -\left[\int_a^{x_0} f - \int_a^{x_n} f\right] = -\int_{x_n}^{x_0} f$$

at which point we then have

$$|F(x_n) - F(x_0)| = \left| -\int_{x_n}^{x_0} f \right| = \left| \int_{x_n}^{x_0} f \right| \leq |M(x_0 - x_n)| = M\,|x_n - x_0|$$

At this point $\{F(x_n)\} \to F(x_0)$ by the Comparison Lemma.

(b) **Example:**
Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \\ x & \text{if } x \in (1, 2] \end{cases}$$

If we calculate F at each point (see next example for a similar problem worked out in detail) we get:

$$F(x) = \begin{cases} 0 & \text{if } x \in [0, 1] \\ \frac{1}{2}x^2 - \frac{1}{2} & \text{if } x \in (1, 2] \end{cases}$$

By the theorem this is continuous.

(c) **Note:** This theorem says nothing about the use of F, just that it is a continuous function!

4. **The Second Fundamental Theorem of Calculus**

(a) **The Second Fundamental Theorem of Calculus:** Suppose the function $f : [a,b] \to \mathbb{R}$ is continuous. Then the function F defined by $F(x) = \int_a^x f$ is an antiderivative of f. In other words

$$\frac{d}{dx}\left[\int_a^x f\right] = f \text{ for all } x \in (a,b)$$

Proof: The previous theorem shows that F is continuous. What we need to do now is to show that for any $x_0 \in (a,b)$ that we have $F'(x_0) = f(x_0)$.
Suppose $\{x_n\} \to x_0$ with $\{x_n\}$ in $(a,b) - \{x_0\}$. We claim that

$$\left\{\frac{F(x_n) - F(x_0)}{x_n - x_0}\right\} \to f(x_0)$$

Then for any n if $x_n > x_0$ we have

$$F(x_n) - F(x_0) = \int_a^{x_n} f - \int_a^{x_0} f = \int_{x_0}^{x_n} f$$

and then the MVT4I ensures we can choose c_n between x_0 and x_n satisfying

$$F(x_n) - F(x_0) = \int_{x_0}^{x_n} f = f(c_n)(x_n - x_0)$$

On the other hand if $x_n < x_0$ we have

$$F(x_n) - F(x_0) = -\left[F(x_0) - F(x_n)\right] = -\left[\int_a^{x_0} f - \int_a^{x_n} f\right] = -\left[\int_{x_n}^{x_o} f\right]$$

and then the MVT4I ensures we can choose c_n between x_0 and x_n satisfying

$$F(x_n) - F(x_0) = -\left[\int_{x_n}^{x_0} f\right] = -\left[f(c_n)(x_0 - x_n)\right] = f(c_n)(x_n - x_0)$$

Note that these two results are exactly the same and can be rewritten as

$$f(c_n) = \frac{F(x_n) - F(x_0)}{x_n - x_0}$$

Lastly since $\{x_n\} \to x_0$ and since c_n is between x_0 and x_n we have $\{c_n\} \to x_0$ and since f is continuous we finally have

$$\left\{\frac{F(x_n) - F(x_0)}{x_n - x_0}\right\} = \{f(c_n)\} \to f(x_0)$$

(b) **Example:** Consider the function $f : [0, 2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ x & \text{if } x \in (1, 2] \end{cases}$$

Consider the function $F : [0, 2] \to \mathbb{R}$ defined by

$$F(x) = \int_0^x f$$

For $x_0 \in [0, 1]$ we apply the FTOC1 to get

$$F(x_0) = \int_0^{x_0} f = \int_0^{x_0} 1 = x\Big|_0^{x_0} = x_0$$

and for $x_0 \in (1, 2]$ we apply the additivity of the integral along with the FTOC1 to get

$$F(x_0) = \int_0^{x_0} f = \int_0^1 1 + \int_1^{x_0} x = x\Big|_0^1 + \frac{1}{2}x^2\Big|_1^{x_0} = 1 + \left[\frac{1}{2}x_0^2 - \frac{1}{2}(1)^2\right] = \frac{1}{2} + \frac{1}{2}x_0^2$$

By the theorem the function

$$F(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ \frac{1}{2} + \frac{1}{2}x^2 & \text{if } x \in (1, 2] \end{cases}$$

is an antiderivative of f.

(c) **Note:** Realize the significant difference in the previous two theorems. The first states that integrability of f is enough to construct a continuous function $F(x) = \int_a^x f$ but it says nothing about the usefulness of this F. It's only when f is continuous that this F is guaranteed to be an antiderivative of f.

5. **Miscellaneous Corrolaries**

(a) **Corrolary:** Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then for all $x \in (a, b)$ we have:

$$\frac{d}{dx}\left[\int_x^b f\right] = -f(x)$$

Proof: Omit.

(b) **Definition:** Suppose $f : [a, b] \to \mathbb{R}$ is integrable. We define:

$$\int_b^a f = -\int_a^b f$$

(c) **Corollary:** Suppose I is an open interval and $f : I \to \mathbb{R}$ is continuous. Suppose $x_0 \in I$ is fixed. Then for all $x \in I$ we have

$$\frac{d}{dx}\left[\int_{x_0}^x f\right] = f(x)$$

Proof: Omit.

(d) **Corollary:** Suppose I is an open interval and $f : I \to \mathbb{R}$ is continuous. Suppose J is an open interval and $\phi : J \to \mathbb{R}$ is differentiable and $\phi(J) \subseteq I$. Suppose $x_0 \in I$ is fixed. Then for all $x \in J$ we have

$$\frac{d}{dx}\left[\int_{x_0}^{\phi(x)} f\right] = f(\phi(x))\phi'(x)$$

Proof: Omit.