Math 410 Section 8.1: Taylor Polynomials

1. **Introduction:** Polynomials are the simplest kind of functions, very easy to work with. Consequently it's interesting to look at what sorts of functions may be approximated by polynomials. First we need to set down a concept of functions being close to one another at a point.

2. **Order of Contact**

   (a) **Definition:** Let I be a neighborhood of x_0. Two functions $f, g : I \to \mathbb{R}$ have contact of order n at x_0 if $f(x_0) = g(x_0)$, $f'(x_0) = g'(x_0)$, ..., and $f^{(n)}(x_0) = g^{(n)}(x_0)$. For example functions have order $n = 0$ if they meet, they have order $n = 1$ if they meet and their first derivatives are equal there, and so on.

   (b) **Example:** Examine the functions $f(x) = x^2$ and $g(x) = x^3$ at $x_0 = 0$. Note that $f(0) = 0$ and $g(0) = 0$, $f'(0) = 0$ and $g'(0) = 0$, but $f''(0) = 2$ whereas $g''(0) = 0$. Thus f and g have contact of order 1 (and 0 in fact) at $x_0 = 0$ but not order 2 or higher.

3. **Taylor Polynomials**

   (a) **Theorem:** Let I be a neighborhood of x_0 and let $n \in \mathbb{N}$. Suppose $f : I \to \mathbb{R}$ has n derivatives. Then there is a unique polynomial of degree at most n which has contact of order n with f at x_0 and that polynomial is given by the formula:

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

   Proof: Suppose we have the polynomial

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ... + a_n(x - x_0)^n = \sum_{k=0}^{n} a_k(x - x_0)^k$$

   Observe that if p_n and f have contact of order n at x_0 then:

   - $f(x_0) = p_n(x_0) = a_1$
   - $p_n'(x) = a_1 + 2a_2(x - x_0) + ... + na_n(x - x_0)^{n-1}$ so $f'(x_0) = p_n'(x_0) = a_1$ so $a_1 = f'(x_0)$
   - $p_n''(x) = 2a_2 + 3!(x - x_0)... + n(n-1)a_n(x - x_0)^{n-2}$ so $f''(x_0) = p_n''(x_0) = 2a_2$ so $a_2 = \frac{f''(x_0)}{2}$
   - $p_n'''(x) = 3!a_3 + 4!(x - x_0)... + n(n-1)(n-2)a_n(x - x_0)^{n-3}$ so $f'''(x_0) = p_n'''(x_0) = 3!a_3$ so $a_3 = \frac{f'''(x_0)}{3!}$
   - and so on and we have our result.

   (b) **Definition:** This polynomial is called the n^{th} Taylor polynomial for f at x_0. In other words

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

   (c) **Critical Note:** It is important to understand that by construction p_n and f only have contact of order n at the single point x_0. The construction says absolutely nothing about how the polynomials behave away from x_0.

4. **Examples of Taylor Polynomials**

(a) **Example:** Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = e^x$ and $x_0 = 0$. Then since $f^{(k)}(x) = e^x$ for all k and x and so $f^{(k)}(0) = 1$ for all k the first few Taylor polynomials for f at $x_0 = 0$ are:

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= 1 + x \\ p_2(x) &= 1 + x + \frac{x^2}{2} \\ p_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \end{aligned}$$

and in general:

$$p_n(x) = 1 + x + \frac{x^2}{2} + ... + \frac{x^n}{n!}$$

(b) **Example:** Suppose $f : (0, \infty) \to \mathbb{R}$ is given by $f(x) = \sqrt{x}$ and $x_0 = 1$. To find the third Taylor polynomial for f at $x_0 = 1$ we find:

- $f(1) = 1$
- $f'(x) = \frac{1}{2}x^{-1/2}$ and so $f'(1) = \frac{1}{2}$
- $f''(x) = -\frac{1}{4}x^{-3/2}$ and so $f''(1) = -\frac{1}{4}$
- $f'''(x) = \frac{3}{8}x^{-5/2}$ and so $f'''(1) = \frac{3}{8}$

and so

$$\begin{aligned} p_3(x) &= 1 + \frac{1}{2}(x-1) + \frac{-1/4}{2}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \end{aligned}$$

(c) **Example:** Suppose $f : (0, \infty) \to \mathbb{R}$ is given by $f(x) = \frac{1}{x}$ and $x_0 = 2$. To find the fourth Taylor polynomial for f at $x_0 = 2$ we find:

- $f(2) = \frac{1}{2}$
- $f'(x) = -\frac{1}{x^2}$ and so $f'(2) = -\frac{1}{4}$
- $f''(x) = \frac{2}{x^3}$ and so $f''(2) = \frac{1}{4}$
- $f'''(x) = -\frac{6}{x^4}$ and so $f'''(2) = -\frac{3}{8}$
- $f''''(x) = \frac{24}{x^5}$ and so $f''''(2) = \frac{3}{4}$

and so

$$\begin{aligned} p_4(x) &= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1/4}{2}(x-2)^2 + \frac{-3/8}{3!}(x-2)^3 + \frac{3/4}{4!}(x-2)^4 \\ &= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 \end{aligned}$$