Math 410 Section 8.2: The Lagrange Remainder Theorem

1. **Introduction:** We know that the function meets the n^{th} Taylor Polynomial at the point x_0 but we don't know for sure what happens at other points. Intuition and evidence seems to suggest that as n gets larger that the Taylor Polyomial seems to get closer to the function near x_0 and perhaps even at points which are not near x_0.

2. **Reminder: The Function Control Theorem** Let I be an open interval and $n \in \mathbb{N}$ and suppose that $f : I \to \mathbb{R}$ has n derivatives. Suppose also that:

$$f(x_0) = f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0$$

Then for each $x \neq x_0$ in I there is some z strictly between x_0 and x with:

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

3. **The Lagrange Remainder Theorem** Let I be a neighborhood of x_0 and $n \in \mathbb{N}$. Suppose $f : I \to \mathbb{R}$ has $n + 1$ derivatives. Then for each $x \in I$ there is some c strictly between x_0 and x such that:

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{p_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}}_{r_n(x)}$$

The expression $r_n(x)$ is the n^{th} Taylor Remainder.

Proof: For all $x \in I$ define:

$$r_n(x) = f(x) - p_n(x)$$

Since $f(x)$ and $p_n(x)$ have contact of order n at x_0 we know that

$$r_n(x_0) = r_n'(x_0) = ... = r_n^{(n)}(x_0) = 0$$

By the Function Control Theorem (applied to $r_n(x)$ and using $n + 1$ in place of n) there is some c strictly between x_0 and x such that:

$$r_n(x) = \frac{r^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Since

$$r^{(n+1)}(c) = f^{(n+1)}(c) - p_n^{(n+1)}(c)$$

and since p_n is a polynomial of degree n its $(n + 1)^{\text{st}}$ derivative is identically zero and so $r^{(n+1)}(c) = f^{(n+1)}(c)$ and so

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$\mathcal{QED}$