

Math 410 Section 8.3: The Convergence of Taylor Polynomials

1. **Introduction:** Suppose we start with a function $f : I \rightarrow \mathbb{R}$ with I a neighborhood of x_0 such that f has all possible derivatives at x_0 and we construct the Taylor Polynomials $p_0(x), p_1(x), p_2(x), \dots$ for f at x_0 . For each x this results in a sequence $\{p_n(x)\}$. We are interested in the possibility that

$$\{p_n(x)\} \rightarrow f(x)$$

meaning the Taylor polynomials converge to the function at that x . If this is the case then we write:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

where the infinite series on the right is called the Taylor Series for f at x .

Notice that since $r_n(x) = f(x) - p_n(x)$ this will happen precisely when $\{r_n(x)\} \rightarrow 0$.

This is especially useful if it is true for some large collection of x since then essentially the Taylor Series (infinite Taylor polynomial) can be used as a substitute for the function.

Pre-Intuition: For each n we have:

$$r_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!} (x - x_0)^{n+1}$$

where we've been very specific to point out that c_n depends on n .

Notice the denominator is $(n+1)!$ and so $\{r_n(x)\} \rightarrow 0$ precisely when the numerator grows slower than this as n increases. The numerator is composed of two parts, $f^{(n+1)}(c_n)$ and $(x - x_0)^{n+1}$. The latter of these doesn't matter because as n grows eventually the additional terms introduced by $(n+1)!$ will outweigh those introduced by multiplying by more $(x - x_0)$ s. So really it's all about making sure that the derivatives don't grow too quickly as n does. A really basic example would be a function like sine or cosine where the derivatives are all bounded by the same constants.

Example: Take $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sin x$ and set $x_0 = 0$. Consider $x = 2$. We know that for each n there is some c_n strictly between 0 and 2 such that:

$$r_n(2) = \frac{f^{(n+1)}(c_n)}{(n+1)!} (2 - 0)^{n+1}$$

Because all derivatives of f are bound between -1 and 1 we then know:

$$|r_n(2)| \leq \frac{2^{n+1}}{(n+1)!}$$

To formalize the idea that $(x - x_0)^{n+1}$ is outweighed by $(n+1)!$ observe that for $n \geq 4$ we have:

$$\begin{aligned} \frac{2^{n+1}}{(n+1)!} &= \left[\frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \right] \left[\frac{2}{4} \cdot \dots \cdot \frac{2}{n+1} \right] \\ &\leq [2 \cdot 2 \cdot 2] \left[\frac{2}{4} \cdot \dots \cdot \frac{2}{4} \right] \\ &\leq 8 \cdot \left(\frac{1}{2} \right)^{n-2} \end{aligned}$$

so that

$$\{r_n(2)\} \rightarrow 0$$

by the Comparison Lemma and consequently

$$\{p_n(2)\} \rightarrow f(2) = \sin(2)$$

As a consequence we can approximate $\sin(2)$ using $p_n(2)$ for very large n .

2. Taylor Polynomial Convergence Theorem

(a) **Pre-Intuition:** First we'll generalize and formalize the calculation shown in the last example. Then we will point out that as long as the derivatives have reasonable constraint we can fold them into the bound.

(b) **Lemma:** For any $\alpha \in \mathbb{R}$ we have

$$\left\{ \frac{\alpha^n}{n!} \right\} \rightarrow 0$$

Proof: Assume $\alpha > 0$. The case where $\alpha < 0$ just involves some well-placed absolute values and the case where $\alpha = 0$ is obvious.

Fix $k \geq 2\alpha$ and then we have:

$$\begin{aligned} \frac{\alpha^n}{n!} &= \left[\frac{\alpha}{1} \cdot \frac{\alpha}{2} \cdot \dots \cdot \frac{\alpha}{k-1} \right] \left[\frac{\alpha}{k} \cdot \dots \cdot \frac{\alpha}{n} \right] \\ &\leq [\alpha \cdot \alpha \cdot \dots \cdot \alpha \cdot \alpha] \left[\frac{1}{2} \cdot \dots \cdot \frac{1}{2} \right] \\ &\leq \alpha^{k-1} \left(\frac{1}{2} \right)^{n-k+1} \\ &\leq (2\alpha)^{k-1} \left(\frac{1}{2} \right)^n \end{aligned}$$

so that the result follows by the Comparison Lemma.

(c) **Theorem (Taylor Polynomial Convergence Theorem):** Let I be a neighborhood of x_0 and suppose $f : I \rightarrow \mathbb{R}$ has all derivatives. Fix $x \in I$ and suppose there $\exists B, M \in \mathbb{R}^+$ such that $\forall n \in \mathbb{N}$ and $\forall c$ strictly between x_0 and x we have

$$\left| f^{(n)}(c) \right| \leq BM^n$$

Then

$$\{r_n(x)\} \rightarrow 0 \text{ and hence } \{p_n(x)\} \rightarrow f(x)$$

Proof: For each n by the LRT we have some c_n strictly between x_0 and x such that

$$r_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!} (x - x_0)^{n+1}$$

It follows that:

$$|r_n(x)| = \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} (x - x_0)^{n+1} \right| \leq \frac{BM^{n+1}}{(n+1)!} |x - x_0|^{n+1} = B \frac{(M|x - x_0|)^{n+1}}{(n+1)!}$$

So then $\{r_n(x)\} \rightarrow 0$ by the Comparison Lemma and the previous Lemma.

(d) **Example:** Let $f(x) = \cos x$ and $x_0 = 2$. Let $x \in \mathbb{R}$. Observe that for all c strictly between $x_0 = 0$ and $x = 2$ (in fact for all c) we have:

$$\left| f^{(n+1)}(c) \right| \leq 1$$

which satisfies the theorem with $M = 1$ and $B = 1$. Thus $\{p_n(x)\} \rightarrow f(x) = e^x$, specifically:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

(e) **Example:** Let $f(x) = 2^x$ and $x_0 = 0$. Let $x \in \mathbb{R}$.

If $x > 0$ then for $0 < c < x$ and so $2^c < 2^x$ and so:

$$|f^n(c)| = |(\ln 2)^n 2^c| = 2^c (\ln 2)^n \leq 2^x (\ln 2)^n$$

which satisfies the theorem with $B = 2^x$ and $M = \ln 2$.

If $x < 0$ then for $x < c < 0$ and so $2^c < 1$ and so:

$$|f^n(c)| = |(\ln 2)^n 2^c| \leq (\ln 2)^n$$

which satisfies the theorem with $B = 1$ and $M = \ln 2$.

Thus for all x we have $\{p_n(x)\} \rightarrow f(x) = 2^x$.

(f) **Comment:**

The hypotheses are sufficient but not necessary.

(g) **Example:** Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be $f(x) = \frac{1}{x}$ and $x_0 = 1$. Let $x \in \mathbb{R}^+$.

Observe that

$$|f^{(n)}(c)| = \left| \frac{n!}{c^{n+1}} \right|$$

which does not satisfy the hypotheses because of the $n!$.

However for each n we have some c_n strictly between 1 and x with

$$|r_n(x)| = \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} (x-1)^{n+1} \right| = \left| \frac{(n+1)!/c_n^{n+2}}{(n+1)!} (x-1)^{n+1} \right| = \frac{1}{x-1} \left[\frac{x-1}{c_n} \right]^{n+2}$$

This converges for some values of x . For example if $1 < x < 2$ then we have $0 < x-1 < 1$ and $1 < c_n < 2$ and so

$$r_n(x) = \frac{1}{x-1} \left[\frac{x-1}{c_n} \right]^{n+2} \leq 2 [x-1]^{n+2}$$

and so

$$\{r_n(x)\} \rightarrow 0$$

by the Comparison Lemma.