Math 410 Section 8.5: The Cauchy Integral Remainder Theorem

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1. **Introduction:** We know that in general the Taylor polynomial does not equal the function (other than at x_0) and so there is a remainder:

$$f(x) = p_n(x) + r_n(x)$$

   We also have a formula for $r_n(x)$, the Lagrange Remainder Formula. However this is not the only formula for the remainder.

2. **Preliminary Definition:** Suppose $f : [a,b] \to \mathbb{R}$ is integrable. We define:

$$\int_b^a f = -\int_a^b f$$

3. **Integration by Parts:** Supppose $f, g : [a,b] \to \mathbb{R}$ are both continuous and have continuous bounded derivatives on (a,b). Then

$$\int_a^b fg' = fg\Big|_a^b - \int_a^b f'g$$

   Proof: We have:

$$\begin{aligned}
(fg)' &= f'g + fg' \\
fg' &= (fg)' - f'g \\
\int_a^b fg' &= \int_a^b (fg)' - \int_a^b f'g \\
\int_a^b fg' &= fg\Big|_a^b - \int_a^b f'g
\end{aligned}$$

   where the last equality holds by the First Fundamental Theorem of Calculus.

4. **The Cauchy Integral Remainder Theorem:** Let I be a neighborhood of x_0 and let $n \in \mathbb{N}$. Suppose $f : I \to \mathbb{R}$ has $n+1$ derivatives and $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Then for each $x \in I$ we have:

$$r_n(x) = \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^n \, dt$$

   In other words:

$$f(x) = p_n(x) + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^n \, dt$$

   Note: The primary thing to note is that while the Lagrange Remainder Formula depends on an unknown (the c), the Cauchy Integral Remainder Formula does not. It does however depend on an integral that is in many cases impractical to calculate. But on a positive note this integral can be approximated using sums.

Proof: We proceed by induction but we will show both $n = 0$ and $n = 1$ just because they're enlightening:

For the case $n = 0$ observe that the right side of the above equals

$$f(x_0) + \frac{1}{0!}\int_{x_0}^{x} f^{(0+1)}(t)(x-t)^0\,dt = f(x_0) + \left[f(t)\Big|_{x_0}^{x}\right] = f(x)$$

where the First Fundamental Theorem of Calculus is used to evaluate the integral.

For the case $n = 1$ observe that by the First Fundamental Theorem of Calculus we have:

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt$$

We proceed by rewriting the integral and integrating by parts:

$$\begin{aligned}
\int_{x_0}^{x} f'(t)\,dt &= \int_{x_0}^{x} f'(t)\frac{d}{dt}(x-t)\,dt \\
&= -f'(t)(x-t)\Big|_{x_0}^{x} + \int_{x_0}^{x} f''(t)(x-t)\,dt \\
&= -f'(x)(x-x) + f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t)\,dt \\
&= f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t)\,dt
\end{aligned}$$

Thus:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^{x} f''(t)(x-t)\,dt$$

and we have our claim for $n = 1$.

Assume that for n we have:

$$f(x) = p_n(x) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt$$

Observe that:

$$\begin{aligned}
\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt &= \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)\left[\frac{d}{dt}\frac{-1}{n+1}(x-t)^{n+1}\right]\,dt \\
&= -\frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+1)}(t)\left[\frac{d}{dt}(x-t)^{n+1}\right]\,dt \\
&= -\frac{1}{(n+1)!}\left[f^{(n+1)}(t)(x-t)^{n+1}\Big|_{x_0}^{x} - \int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt\right] \\
&= -\frac{1}{(n+1)!}\left[0 - f^{(n+1)}(x_0)(x-x_0)^{n+1} - \int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt\right] \\
&= \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt
\end{aligned}$$

So that

$$\begin{aligned}
f(x) &= p_n(x) + \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt \\
f(x) &= p_{n+1}(x) + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(t)(x-t)^{n+1}\,dt
\end{aligned}$$