Math 410 Section 8.7: The Weierstrass Approximation Theorem

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1. **Theorem (The Weierstrass Approximation Theorem):**

   Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then for all $\epsilon > 0$ there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that for all $x \in [a, b]$ we have $|f(x) - p(x)| < \epsilon$.

2. **Two Lemmas:** There are two Lemmas needed which we will state without proof.

   (a) **Lemma A:** For $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have:

   $$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$$

   (b) **Lemma B:** For $x \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 2$ we have:

   $$\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n}$$

3. **Proof of the WAT:**

   Note: We can focus on the case $[0, 1]$ since for any other $f : [a, b] \to \mathbb{R}$ simply define $g(x) = f\left(\frac{x-a}{b-a}\right)$, apply the WAT to $g : [0, 1] \to \mathbb{R}$ to get $q(x)$, then define $p(x) = q((b - a)x + a)$. All we're doing here is vertically shifting and stretching/shrinking the function so it's on $[0, 1]$, applying the $[0, 1]$ version and then shrinking/stretching back. Since this does not affect the y-values the theorem holds.

   Thus given $f : [0, 1] \to \mathbb{R}$ and $\epsilon > 0$ choose each of the following:

   - Since f is continuous on $[0, 1]$ it is uniformly continuous on $[0, 1]$. Choose $\delta > 0$ such that for all $u, v \in [0, 1]$ we have $|u - v| < \delta$ implies $|f(u) - f(v)| < \frac{\epsilon}{2}$.
   - By the Extreme Value Theorem choose M so that for all $x \in [0, 1]$ we have $|f(x)| \leq M$.
   - Choose $n \in \mathbb{N}$ so that $n > \frac{4M}{\epsilon\delta^2}$ and notice that this tells us: **Fact C**: We have $\frac{2M}{n\delta^2} < \frac{\epsilon}{2}$.

   Define

   $$p(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

   We claim that $p(x)$ does the job.

   Let $x \in [a, b]$ and let $1 \leq k \leq n$. Either $\left|x - \frac{k}{n}\right| < \delta$ or $\left|x - \frac{k}{n}\right| \geq \delta$.

   - If $\left|x - \frac{k}{n}\right| < \delta$ then $\left|f(x) - f\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2}$.
   - If $\left|x - \frac{k}{n}\right| \geq \delta$ then $\left|f(x) - f\left(\frac{k}{n}\right)\right| \leq 2M$ by the EVT above and since $\left|x - \frac{k}{n}\right| \geq \delta$ we have $\left(x - \frac{k}{n}\right)^2 \geq \delta^2$ and hence $\frac{1}{\delta^2}\left(x - \frac{k}{n}\right)^2 \geq 1$ and hence $\left|f(x) - f\left(\frac{k}{n}\right)\right| \leq 2M \leq \frac{2M}{\delta^2}\left(x - \frac{k}{n}\right)^2$.

   Thus in all cases we have $\left|f(x) - f\left(\frac{k}{n}\right)\right| \leq \frac{\epsilon}{2} + \frac{2M}{\delta^2}\left(x - \frac{k}{n}\right)^2$.

Next since

$$f(x) = f(x) \cdot 1 = f(x) \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}}_{\text{Lemma A}} = \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k}$$

We have

$$\begin{aligned} f(x) - p(x) &= \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \left[f(x) - f\left(\frac{k}{n}\right) \right] \binom{n}{k} x^k (1-x)^{n-k} \end{aligned}$$

and so by the Triangle Inequality we have:

$$\begin{aligned} |f(x) - p(x)| &\leq \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \sum_{k=0}^{n} \left[\frac{\epsilon}{2} + \frac{2M}{\delta^2} \left(x - \frac{k}{n} \right)^2 \right] \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \frac{\epsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} + \frac{2M}{\delta^2} \underbrace{\sum_{k=0}^{n} \left(x - \frac{k}{n} \right)^2 \binom{n}{k} x^k (1-x)^{n-k}}_{\text{Lemma B}} \\ &= \frac{\epsilon}{2} + \frac{2M}{\delta^2} \underbrace{\frac{x(1-x)}{n}}_{0 \leq x(1-x) < 1} \\ &< \frac{\epsilon}{2} + \underbrace{\frac{2M}{n\delta^2}}_{\text{Fact C}} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

4. **Example** Consider the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 0.5 \\ 1 & \text{if } 0.5 < x \leq 1 \end{cases}$$

If we want our polynomial to be within 0.1 of $f(x)$ then because of the slope of 2 and because $\frac{\epsilon}{2} = 0.05$ we need to choose $\delta = 0.025$. Observe that $M = 1$ satisfies $|f(x)| < M$ for all $x \in [0,1]$.

We then choose $n > \frac{4M}{\epsilon\delta^2} = \frac{4(1)}{(0.1)(0.025)^2} = 64000$ and we define our polynomial:

$$p(x) = \sum_{k=0}^{64000} f\left(\frac{k}{64000}\right) \binom{k}{64000} x^k (1-x)^{64000-k}$$

Notice this polynomial has degree 64000. Which is funny.