Math 410 Section 9.1: Sequences and Series of Numbers

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1. **Introduction:** The goal of this section is to formally define a series and what it means for a series to converge. In addition we will introduce the concept of a Cauchy sequence and look at news methods for showing that sequences and series converge.

2. **Series**

   (a) **Definition:** Given a sequence $\{a_n\}$ we construct the series (infinite sum) represented by:

   $$\sum_{n=1}^{\infty} a_n$$

   (b) **Convergence of Series** Given a series as above we define the n^{th} partial sum:

   $$s_n = \sum_{k=1}^{n} a_n = a_1 + a_2 + ... + a_n$$

   We then say that the series convergences when the sequence of partial sums

   $$\{s_n\} = \left\{ \sum_{k=1}^{n} a_n \right\}$$

   converges. If it converges to some S then we say that the series converges to S and we write:

   $$\sum_{n=1}^{\infty} a_n = S$$

   Basically we are asking: As we add more and more terms does the resulting sum approach something?

   (c) **Example:** Consider $\sum_{n=1}^{\infty} (1/2)^n$. For each k we have

   $$s_k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} = 1 - \frac{1}{2^k}$$

   and since $\{s_k\} \to 1$ we have $\sum_{n=1}^{\infty} (1/2)^n = 1$.

   (d) **Theorem:** If $|r| < 1$ then $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.
   Proof: This is just like the 1/2 case above but using the formula for a geometric sum to get an expression for the n^{th} partial sum.

   (e) **Example:** Consider $\sum_{n=1}^{\infty} (-1)^n$. Here we have $s_k = -1$ for k odd and $s_k = 0$ for k even. Thus since $\{s_k\}$ diverges we say that $\sum_{n=1}^{\infty} 1$ diverges.

   (f) **Theorem:** Suppose $\sum_{k=1}^{\infty} a_n = S$. Then $\{a_n\} \to 0$.
   Proof: We know $\{s_n\} \to S$ but also $\{s_{n-1}\} \to S$. Then $\{a_n\} = \{s_n - s_{n-1}\} \to S - S = 0$.

3. **Cauchy Sequences**

(a) **Definition:** A sequence $\{a_n\}$ is said to be Cauchy (or to be a Cauchy sequence) if for all $\epsilon > 0$ there exists some N such that if $n_1, n_2 \geq N$ then $|a_{n_1} - a_{n_2}| < \epsilon$.

Note: Intuitively a sequence is Cauchy if, given a closeness, we can give a cutoff after which all elements are within that closeness of one another. It's not usually much easier proving a sequence is Cauchy than proving it converges but it's a handy alternative which appears in theorems.

(b) **Theorem (The Cauchy Convergence Criterion):** A sequence converges iff it is Cauchy.
Proof: Suppose that $\{a_n\} \to a$. Given $\epsilon > 0$ choose N so that if $n \geq N$ then $|a_n - a| < \frac{\epsilon}{2}$. Then if $n_1, n_2 \geq N$ then

$$|a_{n_1} - a_{n_2}| = |(a_{n_1} - a_n) + (a_n - a_{n_2})| \leq |a_{n_1} - a_n| + |a_n - a_{n_2}| < \epsilon$$

Suppose that $\{a_n\}$ is Cauchy. First we'll show that $\{a_n\}$ is bounded. For $\epsilon = 1$ choose N so that if $n_1, n_2 \geq N$ we have $|a_{n_1} - a_{n_2}| < 1$. Specifically then if $n \geq N$ we have $|a_n - a_N| < 1$ and so $-1 < a_n - a_N < 1$ and so $a_N - 1 < a_n < a_N + 1$. Consider $M = \max\{a_1, a_2, ..., a_{N-1}, a_N + 1\}$. We see that for all n we have $a_n \leq M$. Likewise consider $m = \min\{a_1, a_2, ..., a_{N-1}, a_N - 1\}$. We see that for all n we have $a_n \geq m$.

We know every bounded sequence has a convergent subsequence and so $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\} \to a$. We claim $\{a_n\} \to a$. Let $\epsilon > 0$. Since $\{a_n\}$ is Cauchy choose N_1 so that if $n_1, n_2 \geq N_1$ then $|a_{n_1} - a_{n_2}| < \frac{\epsilon}{2}$. Since $\{a_{n_k}\} \to a$ choose N_2 so that if $n_k \geq N_2$ then $|a_{n_k} - a| < \frac{\epsilon}{2}$. Then if $n, n_k \geq \max\{N_1, N_2\}$ then

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \epsilon$$

(c) **Theorem (The Cauchy Convergence Criterion for Series):** The series $\sum_{n=0}^{\infty} a_n$ converges iff for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $k \in \mathbb{N}$ we have

$$|a_{n+1} + ... + a_{n+k}| < \epsilon$$

Proof: By definition the series converges iff $\{s_n\}$ converges which occurs iff it is Cauchy. This is true iff for all $n_1, n_2 \geq N$ we have $|s_{n_1} - s_{n_2}| < \epsilon$. WLOG we can take $n_1 > n_2$ so let $n = n_2$ and $n_1 = n + k$ and then we have

$$\begin{aligned} |s_{n_1} - s_{n_2}| &< \epsilon \\ |s_{n+k} - s_n| &< \epsilon \\ |(a_1 + ... + a_n + a_{n+1} + ... + a_{n+k}) - (a_1 + ... + a_n)| &< \epsilon \\ |a_{n+1} + ... + a_{n+k}| &< \epsilon \end{aligned}$$