Math 410 Section 9.2: Pointwise Convergence of Functions

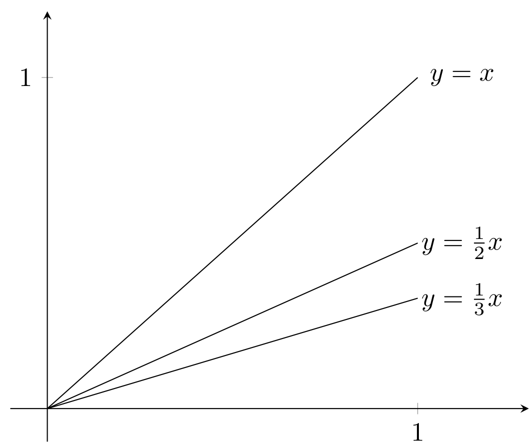
1. **Introduction:** The goal of this section is to introduce the first of two ways in which a sequence of functions can converge to a function and to see what sorts of properties, if any, are preserved.

2. **Definition:** We say that a sequence of functions $\{f_n : D \to \mathbb{R}\}$ and a function $f : D \to \mathbb{R}$ we say that $\{f_n\}$ converges pointwise to f and write if

$$\{f_n\} \underset{p}{\to} f_n \text{ if } \forall x \in D \text{ we have } \{f_n(x)\} \to f(x)$$

   Note 1: This is nonstandard notation.
   Note 2: It is critical to appreciate the fact that the $\forall x \in D$ comes before the $\{f_n(x)\} \to f(x)$. For any particular values of x the convergence is of a sequence and that convergence may be at different rates depending on the x. For example for one particular x_1 the value of $f_n(x_1)$ might be close to $f(x_1)$ for low n but for another x_2 it might take a very large n.

3. **Example - Blah:** Consider $f_n : [0,1] \to \mathbb{R}$ given by $f(x) = \frac{1}{n}x$. This is a line of slope $\frac{1}{n}$ joining $(0,0)$ to $(1, \frac{1}{n})$.

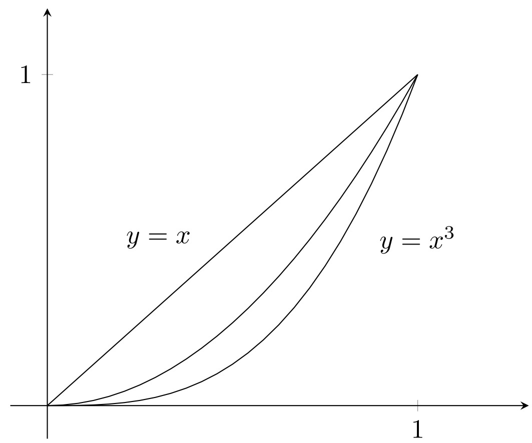


   Here we see intuitively that at each point x the function is approaching 0. More rigorously for each fixed $x \in [0,1]$ we have

$$\{f_n(x)\} = \left\{\frac{1}{n}x\right\} \to 0$$

   and so we write $\{f_n\} \underset{p}{\to} f$.

4. **Example, Destroying Continuity:** Consider $f_n : [0,1] \to \mathbb{R}$ given by $f(x) = x^n$. Here are the first few of these:



Intuitively it appears that for $x \in [0,1)$ we the y-values approach 0 but for $x = 1$ the y-values stays at 1. In fact we can see this rigorously. For $x \in [0,1]$ we have

$$\{f_n(x)\} = \{x^n\} \to 0$$

and for $x - 1$ we have

$$\{f_n(x)\} = \{1\} \to 0$$

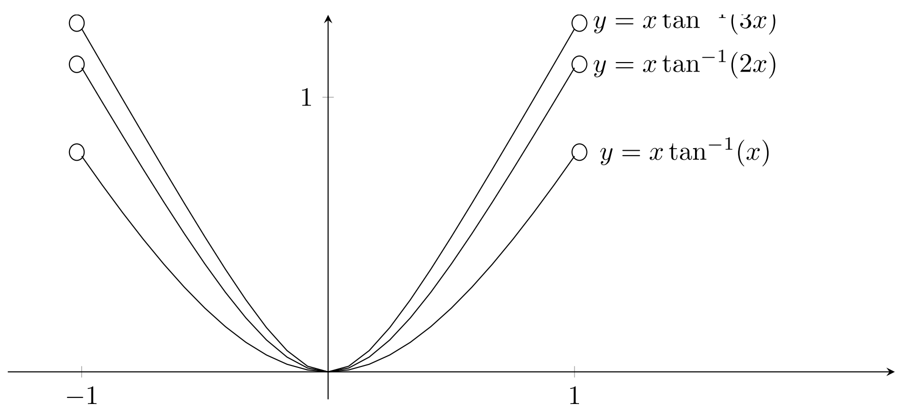
It follows that $\{f_n\} \underset{p}{\to} f$ where:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

This example is particularly interesting because each f_n is continuous on $[0,1]$ but f is not.

It follows that pointwise convergence does not preserve continuity.

5. **Example - Destroying Differentiability:** Consider $f_n : (0,1) \to \mathbb{R}$ given by $f(x) = x\tan^{-1}(nx)$. Here are some of these:



It's not entirely clear what these are approaching so let's look at a specific x.

- Of course if $x = 0$ then $\left\{x\tan^{-1}(nx)\right\} = \{0\} \to 0$.

For other x since $\tan^{-1}$ has horizontal asymptotes at $\pm\frac{\pi}{2}$ we have:

- If $x > 0$ then: $\left\{x\tan^{-1}(nx)\right\} \to x\left(\frac{\pi}{2}\right)$.
- If $x < 0$ then: $\left\{x\tan^{-1}(nx)\right\} \to -x\left(\frac{\pi}{2}\right)$.

So all together:

$$\left\{x\tan^{-1}(nx)\right\} \underset{p}{\to} f(x) = \frac{\pi}{2}|x|$$

This example is interesting because each f_n is continuous and differentiable on $(-1,1)$ and while f is continuous on $(-1,1)$ it is not differentiable on $(-1,1)$.

It follows that pointwise convergence does not preserve differentiability (even when it preserves continuity).

6. **Example - Destroying Integrability:** Since the rationals in $[0,1]$ are countable we can list them all as $\{q_1, q_2, ...\}$. Define $f_n : [0,1] \to \mathbb{R}$ as:

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{q_1, ..., q_n\} \\ 0 & \text{otherwise} \end{cases}$$

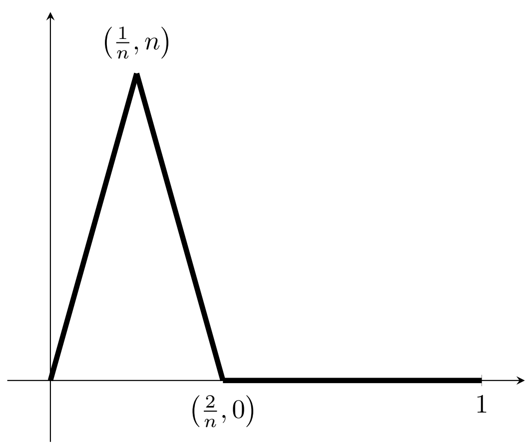
Since each f_n is a step function is it integrable. However $\{f_n\} \underset{p}{\to} f$ where f is:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

which we have seen is not integrable.

It follows that pointwise convergence does not preserve integrability.

7. **Example - Preserving Integrability but Destroying the Integral:** Consider $f_n : [0,1] \to \mathbb{R}$ defined by the following piecewise function:



This is a particularly sneaky example. First note that $f_n(0) = 0$.

Now then for $x \in (0,1]$ as n increases the peak moves left and up but the point $\left(\frac{2}{n}, 0\right)$ (the right edge of the mountain) also moves left. As a result, for any x if we choose a high enough n such that $\frac{2}{n} < x$ then for that n and higher we have $f_n(x) = 0$ (since the mountain has moved to the left of that x). Thus for all x we have $\{f_n(x)\} \to 0$ and so

$$\{f_n\} \underset{p}{\to} f(x) = 0$$

This example is interesting because each f_n is integrable with $\int_0^1 f_n = 1$ (the area of the triangle) and f is also integrable but $\int_0^1 f = 0$.

It follows that pointwise convergence does not preserve the value of the integral even when it preserves integrability.