Math 410 Section 9.3: Uniform Convergence of Functions

1. **Introduction:** We saw in the previous section that pointwise convergence $\{f_n\} \underset{p}{\to} f$ doesn't necessarily preserve continuity, differentiability, integrability or the value of the integral. What we will do now is introduce a stronger form of convergence which will, for the most part, preserve these things.

   Just for a point of review, the definition of pointwise convergence broken down into all its quantifier glory is:

   $$\forall x \in D, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} \text{ if } n \geq N \text{ then } |f_n(x) - f(x)| < \epsilon$$

2. **Definition:** We say that a sequence of functions $\{f_n : D \to \mathbb{R}\}$ and a function $f : D \to \mathbb{R}$ we say that $\{f_n\}$ converges uniformaly to f and write if

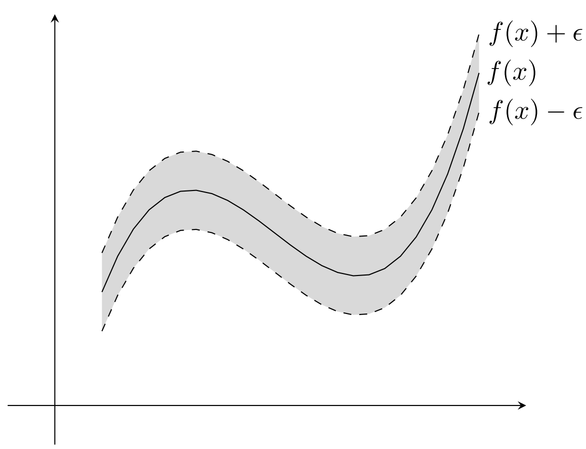
   $$\{f_n\} \underset{u}{\to} f_n$$

   precisely when

   $$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \forall x \in D \text{ if } n \geq N \text{ and if } x \in D, \text{ then } |f_n(x) - f(x)| < \epsilon$$

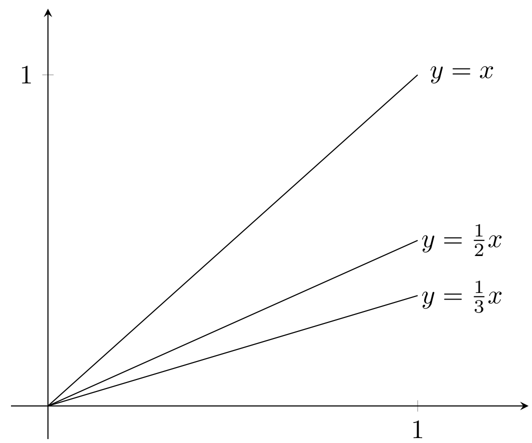
   Note 1: This is nonstandard notation.
   Note 2: The critical difference between pointwise and uniform convergence is that with uniform convergence, given an ϵ, then N cutoff works for all $x \in D$. With pointwise convergence each x has its own N for each ϵ. More intuitively all points on the $\{f_n\}$ are converging together to f.

3. **Visual:** The idea of uniform convergence is helped by a picture illustrating that for any $\epsilon > 0$ we can find an N so that for $n \geq N$ we have all $f_n(x)$ always between the dashed lines:



4. **Examples**

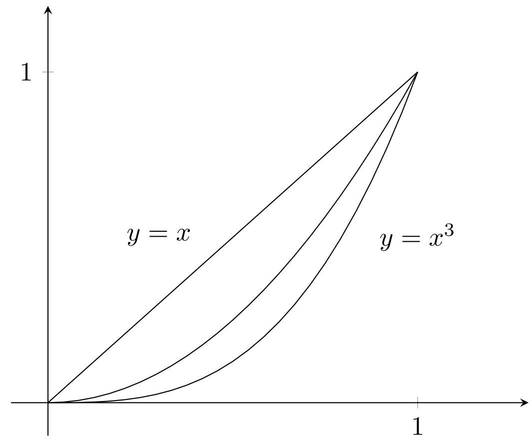
(a) Example: Consider the example from 9.2 where $f_n : [0,1] \to \mathbb{R}$ given by $f(x) = \frac{1}{n}x$. This is a line of slope $\frac{1}{n}$ joining $(0,0)$ to $(1, \frac{1}{n})$.



To see that $\{f_n\} \underset{u}{\to} f$ observe that the maximum difference between $f_n(x) = \frac{x}{n}$ and $f(x) = 0$ (taken over all $x \in [0,1]$) is $1/n$ at the right endpoint. Thus to guarantee $|f_n(x) - f(x)| < \epsilon$ we only need $\frac{1}{n} < \epsilon$ or $n > \frac{1}{\epsilon}$. Thus given $\epsilon > 0$ if we choose $N > \frac{1}{\epsilon}$. Then if $n \geq N$ and if $x \in [0,1]$ then

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right| \leq \frac{1}{n} \leq \frac{1}{N} < \frac{1}{\epsilon}$$

(b) Example: Consider the example from 9.2 where $f_n : [0,1] \to \mathbb{R}$ given by $f(x) = x^n$. Here are the first few of these:



To see that $\{f_n\} \underset{u}{\not\to} f$ observe that intuitively no matter how high n goes there are still x-values with $f(x)$ close to 1, not close to $f(x) = 0$. More rigorously we claim the negation of:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, \forall x \in D \text{ if } n \geq N \text{ and if } x \in D, \text{ then } |f_n(x) - f(x)| < \epsilon$$

In other words we claim:

$$\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, \exists x \in [0,1], n \geq N \text{ and } |x^n - 0| \geq \epsilon$$

Pick $\epsilon = \frac{1}{2}$. For any N let $n = N$ and choose x satisfying $x^n \geq \frac{1}{2}$. by choosing $x \geq \sqrt[n]{\frac{1}{2}}$.

Note that as N gets larger so does our choice of $n = N$ and therefore so does $\sqrt[n]{\frac{1}{2}}$ and hence so does the choice of x, which makes sense, as we have to move further to the right to find points that are $\epsilon = \frac{1}{2}$ away from 0.

5. **A Theorem for Uniform Convergence**

(a) **Introduction:** One issue with proving that a sequence of functions converges uniformly to a target function is that we have to know the target function in advance. This next theorem will find a way around that.

(b) **Definition:** We say a sequence of functions $\{f_n : D \to \mathbb{R}\}$ is uniformly Cauchy if:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, k \in \mathbb{N}, \forall x \in D \text{ if } n \geq N \text{ then } |f_{n+k}(x) - f_n(x)| < \epsilon$$

Basically this is saying that for any ϵ there is a cutoff after which any two of the functions in the sequence are within ϵ of one another at all x.

(c) **Theorem (The Weierstrass Uniform Convergence Criterion):** The sequence of functions $\{f_n : D \to \mathbb{R}\}$ converges uniformly to some $f : D \to \mathbb{R}$ iff the sequence $\{f_n\}$ is uniformly Cauchy.
Proof: Omit for now.

(d) **Example:** This theorem is very useful when it comes to proving the convergence of sequences of functions which themselves are created by sums. For example define $f_n : [-1, 1] \to \mathbb{R}$ by

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{k2^k}$$

Observe that for any $n, k \in \mathbb{N}$ we have:

$$\begin{aligned}
|f_{n+k}(x) - f_n(x)| &= \left| \sum_{k=1}^{n+k} \frac{x^k}{k2^k} - \sum_{k=1}^{n} \frac{x^k}{k2^k} \right| \\
&= \left| \sum_{k=n+1}^{n+k} \frac{x^k}{k2^k} \right| \\
&\leq \sum_{k=n+1}^{n+k} \frac{|x|^k}{k2^k} \\
&\leq \sum_{k=n+1}^{n+k} \frac{1}{k2^k} \\
&\leq \sum_{k=n+1}^{n+k} \frac{1}{2^k} \\
&\leq \frac{1}{2^n} \sum_{k=1}^{k} \frac{1}{2^k} \\
&\leq \frac{1}{2^n}(1)
\end{aligned}$$

It follows that for all $\epsilon > 0$ we only need $\frac{1}{2^n} < \epsilon$ or $2^n > \frac{1}{\epsilon}$. Thus choose N so that $2^N > \epsilon$ and then if $n \geq N$ we satisfy the criteria.

Notice we had to eliminate the k from the expression because while we get to control n by controlling N (since $n \geq N$) the result has to hold for all $k \in \mathbb{N}$.

This is an interesting example because even though we now know that $\{f_n\}$ converges uniformly to some f, we have no real idea what that f is!