Math 410 Section 9.4: Properties of Uniform Convergence

1. **Introduction:** Our hope is that when $\{f_n\} \underset{u}{\to} f$ that some properties are also passed over. We will see that this is true to a degree.

2. **Theorem (Continuity):** Suppose that $\{f_n : D \to \mathbb{R}$ are all continuous, $f : D \to \mathbb{R}$ and and $\{f_n\} \underset{u}{\to} f$. Then f is continuous.

   Proof: Let $x_0 \in D$ and suppose $\{x_k\} \to x_0$. We claim $\{f(x_k)\} \to f(x_0)$. Suppose we are given $\epsilon > 0$.

   - First since $\{f_n\} \underset{u}{\to} f$ we can choose a fixed N so that f_N is within $\frac{\epsilon}{3}$ of f everywhere. Thus $f(x_k)$ is within $\frac{\epsilon}{3}$ of $f_N(x_k)$ and $f(x_0)$ is within $\frac{\epsilon}{3}$ of $f_N(x_0)$.
   - Second since f_N is continuous we can choose K such that if $k \geq K$ then $|f_N(x_k) - f_N(x_0)| < \frac{\epsilon}{3}$.

   Then for $k \geq K$ we have:

$$\begin{aligned}
|f(x_k) - f(x_0)| &= |f(x_k) - f_N(x_k) + f_N(x_k) - f_N(x_0) + f_N(x_0) - f(x_0)| \\
&\leq |f(x_k) - f_N(x_k)| + |f_N(x_k) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

3. **Theorem (Integrability):** Suppose that $\{f_n : [a,b] \to \mathbb{R}$ are all integrable, $f : [a,b] \to \mathbb{R}$ and and $\{f_n\} \underset{u}{\to} f$. Then f is integrable and

$$\left\{ \int_a^b f_n \right\} \to \int_a^b f$$

Proof: Given $\epsilon > 0$ let $\epsilon' = \frac{\epsilon}{2(b-a)}$ since $\{f_n\} \underset{u}{\to} f$ we can choose some $n \in \mathbb{N}$ so that $f_n(x)$ is within ϵ' of $f(x)$ everywhere. That is, for all $x \in [a,b]$ we have:

$$|f(x) - f_n(x)| < \epsilon'$$
$$-\epsilon' < f(x) - f_n(x) < \epsilon'$$
$$f_n(x) - \epsilon' < f(x) < f_n(x) + \epsilon'$$

It follows by the monotonicity of the lower and upper integals (this follows from their definition) that:

$$\overline{\int_a^b} f \le \overline{\int_a^b} [f_n + \epsilon'] = \int_a^b [f_n + \epsilon'] = \left[\int_a^b f \right] + \epsilon'(b-a)$$

and

$$\underline{\int_a^b} f \ge \underline{\int_a^b} [f_n - \epsilon'] = \int_a^b [f_n - \epsilon'] = \left[\int_a^b f \right] - \epsilon'(b-a)$$

Thus

$$\overline{\int_a^b} f - \underline{\int_a^b} f \le \left[\left[\int_a^b f \right] + \epsilon'(b-a) \right] - \left[\left[\int_a^b f \right] - \epsilon'(b-a) \right] = 2\epsilon'(b-a) = \epsilon$$

Since this holds for any $\epsilon > 0$ we must have

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

and hence f is integrable.

Next consider that for any $\epsilon > 0$ let $\epsilon' = \frac{\epsilon}{b-a}$ then since $\{f_n\} \underset{u}{\to} f$ we can choose some $n \in \mathbb{N}$ so that $f_n(x)$ is within ϵ' of $f(x)$ everywhere. As before we have:

$$-\epsilon' < f_n(x) - f(x) < \epsilon'$$
$$\int_a^b \epsilon' < \int_a^b [f_n - f] < \int_a^b \epsilon'$$
$$-\epsilon'(b-a) < \int_a^b f_n - \int_a^b f < \epsilon'(b-a)$$
$$\left| \int_a^b f_n - \int_a^b f \right| < \epsilon'(b-a)$$
$$\left| \int_a^b f_n - \int_a^b f \right| < \epsilon$$

Since this holds for any ϵ we have

$$\left\{ \int_a^b f_n \right\} \to \int_a^b f$$

4. **Theorem (Differentiability):** Let I be an open interval in $\mathbb{R}$. Suppose that $\{f_n : I \to \mathbb{R}\}$ is such that:

- The f_n are all differentiable with continuous derivatives.
- We have $\{f_n\} \underset{p}{\to} f$ for some $f : I \to \mathbb{R}$.
- We have $\{f_n'\} \underset{u}{\to} g$ for some $g : I \to \mathbb{R}$.

Then f is differentiable with a continuous derivative and for all $x \in I$ we have $f'(x) = g(x)$

Note: The fact that it's not sufficient to have $\{f_n\} \underset{u}{\to} f$ with all the f_n differentiable is pointed out by the example in Section 9.2 with $f_n(x) = x\tan^{-1}(nx)$ on $(-1, 1)$. This sequence does converge uniformly to $\frac{\pi}{2}|x|$ and yet differentiability is not carried over to f.

This example doesn't satisfy these hypotheses because even though the sequence of derivative functions converges to some g, that convergence is pointwise and not uniform. The easiest way to see this is to note that for $x > 0$ since $\{f_n(x)\} \to \frac{\pi}{2}x$ we must have $\{f_n'(x)\} \to \frac{\pi}{2}$ but no matter how large n is there are always points close to $x = 0$ with slope arbitrarily close to 0 and therefore far from $\frac{\pi}{2}$.

We can see this by examining the derivative:

$$f_n(x) = \tan^{-1}(nx) + \frac{nx}{1 + n^2x^2}$$

By making x very small we can make both summands small.

Proof: Fix $x_0 \in I$ and note that for each $x \in I$ by the First Fundamental Theorem of Calculus we have:

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} f_n'$$

Now then since $\{f_n'\} \underset{u}{\to} g$ we have:

$$\left\{\int_{x_0}^{x} f_n'\right\} \to \int_{x_0}^{x} g$$

and since $\{f_n\} \underset{p}{\to} f$ we have:

$$\{f_n(x) - f_n(x_0)\} \to f(x) - f(x_0)$$

It follows therefore that:

$$f(x_0) - f(x) = \int_{x_0}^{x} g$$

Since these are identical if the right side is differentiable then so is the left side. And indeed since the f_n' are continuous and $\{f_n'\} \underset{u}{\to} g$ we know that g is continuous and so by the Second Fundamental Theorem of Calculus we have:

$$\frac{d}{dx}\int_{x_0}^{x} g = g$$

and so:

$$\frac{d}{dx}\left[f(x) - f(x_0)\right] = g$$
$$f'(x) = g(x)$$

Note that since $f' = g$ and g is continuous we know f' has continuous derivative and since f is differentiable it must be continuous.