Math 410 Section 9.5: Power Series

1. **Introduction:** We started this chapter by taking a function and creating the series of Taylor Polynomials from the function. Now we will go the other way, we'll start with a series which converges and use it to define a function.

2. **Theorem (Ratio Test):** Consider the series:

$$\sum_{n=0}^{\infty} a_n$$

Suppose that:

$$\left\{ \left| \frac{a_{n+1}}{a_n} \right| \right\} \to L$$

Then:

- If $L < 1$ then the series converges (absolutely).
- If $L > 1$ then the series diverges.

Proof: Omit.

3. **Definition:** Given a sequence $\{c_n\}$ we define the domain of convergence of the series:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ...$$

to be the set D of all $x \in \mathbb{R}$ such that the series converges. Note that D is nonempty since $0 \in D$. Then we can define $f : D \to \mathbb{R}$ by:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

and we say that the series is a power series expansion of $f(x)$.

Example: Consider the series:

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$$

We define:

$$a_n = \frac{2^n x^n}{n+1}$$

and observe that:

$$\left\{ \left| \frac{a_{n+1}}{a_n} \right| \right\} = \left\{ \left| \frac{2^{n+1} x^{n+1}}{(n+2)} \cdot \frac{n+1}{2^n x^n} \right| \right\} = \left\{ 2|x| \cdot \frac{n+1}{n+2} \right\} \to 2|x|$$

It follows that the series converges (absolutely) when $2|x| < 1$ which is when $|x| < \frac{1}{2}$.

4. **Theorem:**

   Consider the power series:

$$\sum_{n=0}^{\infty} c_n x^n$$

   If $r \neq 0$ is in the domain of convergence of the power series then so is the entire interval $(-|r|, |r|)$.

   In addition the power series converges uniformly on this interval.

   Proof: Omit.

   Meaning: For example if $x = 5$ is in the domain of convergence then the domain of convergence contains all of $(-5, 5)$.

   Corollary: The domain of convergence of a power series always has one of the forms $\{0\}$, (r, r), $[r, r)$, $(r, r]$ or $[r, r]$. We can have $r = \infty$ in the parenthetical cases.

5. **Theorem (Differentiation):**

   Consider the power series:

$$\sum_{n=0}^{\infty} c_n x^n$$

   Suppose $(-r, r)$ is in the domain of convergence then the function $f : (-r, r) \to \mathbb{R}$ defined by this power series has derivatives of all order and the derivatives may be calculated on a term-by-term basis. In other words:

$$\frac{d^n}{dx^n} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{d^n}{dx^n} \left[c_n x^n \right]$$

   And in fact all of these derivatives also converge on $(-r, r)$.

   Proof: Omit.

   Example: The earlier example yielded $f : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$ defined by:

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

   It follows that the function f is differentiable and

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \left[\frac{2^n x^n}{n!} \right]$$

   Notice we need to be careful if we rewrite this as a sum because the 0^{th} term vanishes as it's constant. The result is therefore:

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \left[\frac{2^n x^n}{n!} \right] = \sum_{n=1}^{\infty} \frac{2^n n x^{n-1}}{n!}$$

6. **Differential Equations:** Functions defined through power series can be useful when dealing with differential equations. Custom-construction of power series to solve differential equations is beyond the scope of the course but we can at the very least consider the following.

Example: Consider the power series:

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = \frac{1}{(3(0))!} + \frac{x^{3(1)}}{(3(1))!} + \frac{x^{3(2)}}{(3(2))!} + ... = 1 + \frac{1}{6}x^3 + \frac{1}{720}x^6 + ...$$

The Ratio Test shows that the power series converges for all x and so defines a function $f : \mathbb{R} \to \mathbb{R}$. It follows that:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \\
f'(x) &= \sum_{n=1}^{\infty} \frac{3nx^{3n-1}}{(3n)!} = \sum_{n=1}^{\infty} \frac{x^{3n-1}}{(3n-1)!} \\
f''(x) &= \sum_{n=1}^{\infty} \frac{(3n-1)x^{3n-2}}{(3n-1)!} = \sum_{n=1}^{\infty} \frac{x^{3n-2}}{(3n-2)!} \\
f'''(x) &= \sum_{n=1}^{\infty} \frac{(3n-2)x^{3n-3}}{(3n-2)!} = \sum_{n=1}^{\infty} \frac{x^{3n-3}}{(3n-3)!} = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = f(x)
\end{aligned}$$

It follows that this $f(x)$ satisfies the differential equation:

$$f'''(x) = f(x)$$

This is interesting because we are familiar with a function which equals its derivative (for example $y = e^x$) and its second derivative (for example $y = e^{-x}$) and even its fourth derivative (for example $y = \sin x$ and $y = \cos x$) but not its third derivative.

Note: In this example $f(0) = 1$. Since any multiple of f also satisfies this differential equation we can multiple by the power series by any c to force $f(0) = c$. For example the function defined by the power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{17x^{3n}}{(3n)!}$$

satisfies the same differential equation and has $f(0) = 17$.

Note: In this example $f'(0) = 0$. Changing this is tricker. One approach is to take a term-by-term antiderivative of f which will then satisify the same differential equation and adjust it accordingly. For example to get $f(0) = 17$ and $f'(0) = 42$ we could do:

$$f(x) = \sum_{n=0}^{\infty} \left[\frac{17x^{3n}}{(3n)!} + \frac{42x^{3n+1}}{(3n+1)!} \right]$$

Observe that the way we have written this is not in a standard power-series way but it's possible to rewrite it.