MATH 310: Groupwork 6 Solutions **Monday 7/16/2012**

1. Distribute the negation signs for each of the following, adjusting other symbols accordingly.

   (a) $\sim(\forall x, P(x)) \equiv$
   Solution: $\sim(\forall x, P(x)) \equiv \exists x, \sim P(x)$

   (b) $\sim(\exists x, Q(x)) \equiv$
   Solution: $\sim(\exists x, Q(x)) \equiv \forall x \sim Q(x)$

   (c) $\sim(\forall x, \exists y, P(x,y) \vee Q(x,y)) \equiv$
   Solution: $\sim(\forall x, \exists y, P(x,y) \vee Q(x,y)) \equiv \exists x, \forall y, (\sim P(x,y) \wedge \sim Q(x,y))$

   (d) $\sim(\exists x, \forall y, P(x,y) \wedge Q(x,y)) \equiv$
   Solution: $\sim(\exists x, \forall y, P(x,y) \wedge Q(x,y)) \equiv \forall x, \exists y, (\sim P(x,y) \vee \sim Q(x,y))$

   (e) $\sim(\exists x, \forall y, P(x,y) \to Q(x,y)) \equiv$
   Solution: $\sim(\exists x, \forall y, P(x,y) \to Q(x,y)) \equiv \forall x, \exists y, (P(x,y) \wedge \sim Q(x,y))$

2. Negate the following sentences, writing the results in full sentences with as few symbols as the original.

   (a) There is an $x \in \mathbb{R}$ so that for all $y \in \mathbb{Z}$ we have $x = y^2$
   Result: For every $x \in \mathbb{R}$ there is some $y \in \mathbb{Z}$ with $x \neq y^2$.

   (b) For every year there is at least one day when it's sunny.
   Result: There is a year for which all days are not sunny.

   (c) For every week there is at least one day where it rains or snows.
   Result: There is a week where on every day it doesn't rain and it doesn't snow.

3. Each of the following is either trivially or vacuously true or both. Determine which and explain why.

   (a) Let $x \in \mathbb{R}$. Prove if $x^2 - x + 10 = 0$ then $x^3 - 7 = 0$.
   Result: Vacuously true since $x^2 - x + 10$ never equals 0 for $x \in \mathbb{R}$.

   (b) Let $x \in \mathbb{R}$. Prove if $-|x+1| > 3$ then $x^2 \geq 0$.
   Result: Booth vacuously and trivially true. Vacuously true since $-|x+1|$ is always ≤ 0 and trivially true since x^2 is always nonnegative.

   (c) Let $n \in \mathbb{Z}$. Prove if $n^2 > 5$ then $|n+1| \geq 0$.
   Result: Trivially true since for any $n \in \mathbb{Z}$ we have $|n+1| \geq 0$.

4. Provide direct proofs of each of the following:

   (a) If $x \in \mathbb{R}$ and $x^2 + x - 12 = 0$ then $x = 3$ or $x = -4$.
   Proof: If $x^2 + x - 12 = 0$ then $(x+4)(x-3) = 0$ so then $x + 4 = 0$ or $x - 3 = 0$ so $x = -4$ or $x = 3$.

   (b) If $n \in \mathbb{Z}$ is even then $3n - 5$ is odd.
   Proof: If n is even then $n = 2k$ for some $k \in \mathbb{Z}$. Then $3n-5 = 3(2k)-5 = 6k-5 = 2(3k-3)+1 = 2m+1$ for $m = 3k-3 \in \mathbb{Z}$ and so $3n-5$ is odd.

   (c) If $n, m \in \mathbb{Z}$ are odd then $mn^2 - n + m$ is odd.
   Proof: If n and m are odd then $n = 2k+1$ and $m = 2l+1$ for $k, l \in \mathbb{Z}$. Then $mn^2 - n + m = (2l+1)(2k+1)^2 - (2k+1) + (2l+1) = 2(4k^2l + 4kl + l + 2k^2 + 2k - k + l) + 1 = 2c + 1$ for $c = 2k^2l + 4kl + l + 2k^2 + 2k - k + l \in \mathbb{Z}$ so $mn^2 - n + m$ is odd.