MATH 310: Homework 1 Solutions

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1. Rewrite each of the following as a readable sentence following standard mathematical practice, correcting any errors and clarifying.

   (a) $x^2 - 3x = 4$, $x \neq -1$, $x = 4$.
   Solution: If $x^2 - 3x = 4$ and $x \neq -1$ then $x = 4$.

   (b) We know n is a multiple of 2, e.g. n is even.
   Solution: We know n is a multiple of 2, i.e. n is even.

   (c) $xy = yx$ for all x and y.
   Solution: For all x and y in $\mathbb{R}$ we have $xy = yx$.

2. Which of the following are sets? For each which is a set give the cardinality. If not write N/A.

   (a) $\mathbb{Z}$
   Solution: ∞

   (b) $\{\mathbb{Z}\}$
   Solution: 1

   (c) $\mathbb{Z}, \mathbb{R}$
   Solution: N/A

   (d) $\emptyset$
   Solution: 0

   (e) $\{\emptyset\}$
   Solution: 1

   (f) $\{..., -4, -2, 0, 2, 4, ...\}$
   Solution: ∞

   (g) $\{1, 2, \emptyset, \{2\}\}$
   Solution: 4

3. Let $S = \{0, 2, 4, 6, 8, 10\}$. Describe each of the following sets as $\{f(x) \mid x \in S \text{ and } p(x)\}$ where $f(x)$ is a function (maybe just x) and $p(x)$ is some condition (maybe no condition) on x. There may be more than one way to do each so try to be as elegant as possible.

   (a) $\{0, 1, 2, 3, 4\}$
   Solution: $\{x/2 \mid x \in S \text{ and } x \leq 8\}$

   (b) $\{0, 4, 8\}$
   Solution: $\{2x \mid x \in S \text{ and } x \leq 4\}$

   (c) $\{0, 4\}$
   Solution: $\{2x \mid x \in S \text{ and } x \leq 2\}$

   (d) $\{13, 19, 25, 31\}$
   Solution: $\{3x + 13 \mid x \in S \text{ and } x \leq 6\}$

4. Explicitly list the elements using non-conditional {} notation in each of the following sets. You may or may not need ellipses.

   (a) $\{2n \mid n \in \mathbb{Z}\}$
   Solution: $\{..., -6, -4, -2, 0, 2, 4, 6, ...\}$

   (b) $\{-n \mid n \in \mathbb{N}\}$
   Solution: $\{..., -3, -2, -1\}$

   (c) $\{5 - n/2 \mid n \in \mathbb{Z} \text{ and } n > 7\}$
   Solution: $\{..., -2, -1.5, -1, -0.5, 0, -0.5, 1\}$

5. (a) If $A = \{1, 2\}$ find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$.
   Solution: $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $|\mathcal{P}(A)| = 4$.

   (b) If $A = \{1, 2, 3, 4\}$ find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$.
   Solution: $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$
   $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ and $|\mathcal{P}(A)| = 16$.

   (c) Suppose $A = \{1, 2, 3, ..., n\}$. Find (and explain how you found) a formula for $|\mathcal{P}(A)|$.
   Solution: $|\mathcal{P}(A)| = 2^n$ is the simplest formula. When we're picking a subset from A we have to make a choice (yes or no) as to whether to pick each element. Since there are n such choices there are 2^n subsets.

6. Find $\mathcal{P}(\emptyset)$, $\mathcal{P}(\mathcal{P}(\emptyset))$ and $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$.
   Solution:
   $\mathcal{P}(\emptyset) = \{\emptyset\}$
   $\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
   $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

7. Give examples of three sets A, B and C satisfying each of:

   (a) $A \subseteq B \subset C$
   Solution: $A = \{1\}$, $B = \{1, 2\}$ and $C = \{1, 2, 3\}$

   (b) $A \in B$, $B \in C$ and $A \notin C$
   Solution: $A = \{1\}$, $B = \{\{1\}, 2\}$ and $C = \{\{\{1\}, 2\}, 3\}$

   (c) $A \in B$ and $A \subset C$.
   Solution: $A = \{1\}$, $B = \{\{1\}, 2\}$ and $C = \{1, 2\}$

8. Let $U = \{1, 2, 3, 4, ..., 20\}$ be the universal set and let $A = \{x \mid x \in U \text{ and } x \text{ is prime}\}$ and $B = \{x \mid x \in U \text{ and } x \text{ is even}\}$. List the elements of each of the following:

   (a) $A \cup B$
   Solution: $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19\}$

   (b) $A \cap B$
   Solution: $A \cap B = \{2\}$

   (c) $\bar{A} \cup B$
   Solution: $\bar{A} \cup B = \{1, 2, 4, 6, 9, 10, 12, 14, 15, 16, 18, 20\}$

   (d) $A \cap \bar{B}$
   Solution: $A \cap \bar{B} = \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$

   (e) $\overline{A \cup B}$
   Solution: $\overline{A \cup B} = \{1, 9, 15\}$

   (f) $\bar{A} - B$
   Solution: $\bar{A} - B = \{1, 9, 15\}$

9. Give examples of two sets A and B such that $|A - B| = |A \cap B| = |B - A| = 3$.
Solution: $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{4, 5, 6, 7, 8, 9\}$

10. Let $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$. List the elements in each of the following sets:

   (a) $\emptyset \cap A$
   Solution: $\emptyset \cap A = \emptyset$

   (b) $\{\emptyset\} \cap A$
   Solution: $\{\emptyset\} \cap A = \{\emptyset\}$

   (c) $\{\emptyset, \{\emptyset\}\} \cap A$
   Solution: $\{\emptyset, \{\emptyset\}\} \cap A = \{\emptyset, \{\emptyset\}\}$

   (d) $\emptyset \cup A$
   Solution: $\emptyset \cup A = A$

   (e) $\{\emptyset\} \cup A$
   Solution: $\{\emptyset\} \cup A = A$

   (f) $\{\emptyset, \{\emptyset\}\} \cup A$
   Solution: $\{\emptyset, \{\emptyset\}\} \cup A = A$

11. Find a collection of closed intervals $S_n = [?, ?]$ enumerated by the natural numbers with

$$\bigcup_{n \in \mathbb{N}} S_n = (-1, 1)$$

Solution: The classic example is $\left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$

12. Identify where each of the following series of logical steps breaks down and explain why. Your explanations may be informal but use full sentences.

   (a) If A and B are sets and $x \in A$ or $x \in B$ then $x \in A \cap B$.
   Solution: The final "then" step breaks down because we could have $x \in A$ but $x \notin B$ in which case $x \notin A \cap B$.

   (b) If $n \in \mathbb{Z}$ and $n \not< 3$ then $n > 3$.
   Solution: The final "then" step breaks down because $n \not< 3$ only gives us $n \geq 3$.

   (c) If $x = 1$ then $x^2 = x$ so $x^2 - 1 = x - 1$ from whence we have $(x - 1)(x + 1) = x - 1$ and so $x + 1 = 1$ giving us finally $1 + 1 = 1$.
   Solution: The step from $(x - 1)(x + 1) = x - 1$ to $x + 1 = 1$ fails because we're dividing by $x - 1$ which equals 0 (because $x = 1$) and division by 0 is not permissible.