MATH 310: Homework 2 Solutions

1. Categorize the following sentences as closed sentences (statements), open sentences or neither. For open sentences list (in nonconditional set notation) the elements in the domain for which the open sentence becomes a true statement.

   (a) This sentence is false.
   Solution: Neither.

   (b) Are there infinitely many twin primes?
   Solution: Neither.

   (c) There are infinitely many prime numbers.
   Solution: Closed sentence.

   (d) $\frac{1}{2}$ is the square root of an integer.
   Solution: Closed sentence.

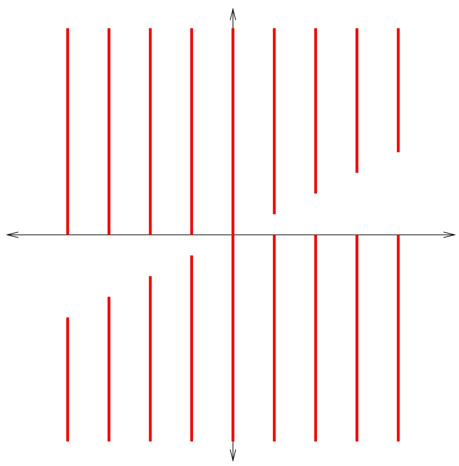
   (e) $x+1$ is divisible by 3 over the domain $S = \{0, 1, 2, 3, ..., 15\}$
   Solution: Open sentence, true for $\{2, 5, 8, 11, 14\}$.

   (f) $A \subseteq \{1, 2\}$ over the domain $S = \mathcal{P}(\{1, 2, 3\})$.
   Solution: Open sentence, true for $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

2. Sketch the subset of $\mathbb{Z} \times \mathbb{R}$ given by

$$\{(x, y) \mid 2y^2 - xy \geq 0\}$$

Solution: The figure is as follows. The implicit diagonal line is $y = \frac{1}{2}x$.



3. For the sets $A = \{1, 2\}$ and $B = \mathcal{P}(\{1, 2, 3\})$ and statements $P : A \in B$ and $Q : A \cap B = B$ fill in the following table:

Statement	T or F
P	T
Q	F
$P \vee Q$	T
$P \wedge Q$	F
$\sim ((\sim P) \vee Q)$	T
$P \to Q$	F
$Q \to P$	T
$P \leftrightarrow Q$	F

4. Let $S = \{1, 2, 3, 4, 5, 6\}$ and consider the open sentences

$$P(A) : A \cup \{1, 2, 3\} = S \text{ and } Q(A) : (1 \notin A) \wedge (2 \notin A)$$

over the domain $\mathcal{P}(S)$. Determine all $A \in \mathcal{P}(S)$ for which $P(A) \wedge Q(A)$ is true.
Solution: Just to be clear, condition $P(A)$ is true if A contains all of $4, 5, 6$ and condition $Q(A)$ is true if A contains neither of 1 or 2.

Thus $P(A) \wedge Q(A)$ is true for $A \in \{\{3, 4, 5, 6\}, \{4, 5, 6\}\}$

5. Repeat the previous exercise with $P(A) \vee Q(A)$.
Solution: The condition $P(A) \vee Q(A)$ is true for

$A \in \{\{4, 5, 6\}, \{1, 4, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}, \{1, 2, 4, 5, 6\},$
$\{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\},$
$\emptyset, \{3\}, \{4\}, \{5\}, \{6\}, \{3, 4\}, \{3, 5\}, \{3, 6\},$
$\{4, 5\}, \{4, 6\}, \{5, 6\}, \{3, 4, 5\}, \{3, 4, 6\}, \{3, 5, 6\}\}.$

6. Suppose $P(x) : x \geq 0$ and $Q(x) : (x + 1)(x - 2) < 0$ over the domain $\mathbb{R}$.

   (a) For which values in the domain is the conditional $P(x) \to Q(x)$ a true statement?
   Solution: We know $P(x) \to Q(x)$ is true when $(\sim P(x)) \vee Q(x)$. Since $\sim P(x)$ is true for $x \in (-\infty, 0)$ and $Q(x)$ is true for $x \in (-1, 2)$ we have $P(x) \to Q(x)$ true for $x \in (-\infty, 2)$.

   (b) For which values in the domain is the conditional $Q(x) \to P(x)$ a true statement?
   Solution: We know $Q(x) \to P(x)$ is true when $(\sim Q(x)) \vee P(x)$. Since $\sim Q(x)$ is true for $x \in (-\infty, -1] \cup [2, \infty)$ and $P(x)$ is true for $x \in [0, \infty)$ we have $Q(x) \to P(x)$ true for $x \in (-\infty, -1] \cup [0, \infty)$.

   (c) For which values in the domain is the biconditional $P(x) \leftrightarrow Q(x)$ a true statement?
   Solution: We need either $P(x) \wedge Q(x)$ which occurs when $x \in [0, 2)$ or $(\sim P(x)) \vee (\sim Q(x))$ which occurs when $x \in (-\infty, -1]$. Together therefore we see $P(x) \leftrightarrow Q(x)$ is true for $x \in (\infty, -1] \cup [0, 2)$.