MATH 310: Homework 3 **Due Thursday 7/19/2012**

1. Suppose $P(A) : A \cap \{1,3\} \neq \emptyset$ and $Q(A) : |A - \{1\}| = 2$. For which $A \in \mathcal{P}(\{1,2,3,4\})$ is the biconditional $P(A) \leftrightarrow Q(A)$ a true statement? Justify your steps, don't just give the answer.

2. For statements P and Q show that $((P \to Q) \wedge (Q \to R)) \to (P \to R)$ is a tautology.

3. Determine with justification if the following are true or false.

   (a) $\forall n \in \mathbb{Z}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.

   (b) $\exists n \in \mathbb{Z}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.

   (c) $\exists! n \in \mathbb{Z}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.

   (d) $\exists! n \in \{0,1,2,3,4\}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.

   (e) $\forall x \in \mathbb{R}$, $x^2 + 3 \geq 0$.

   (f) $\exists x \in \mathbb{R}$, $x^2 + 3 \geq 0$.

   (g) $\forall x \in \{1,2,3\}$, $3x + 1$ is prime.

   (h) $\exists x \in \{1,2,3\}$, $3x + 1$ is prime.

   (i) $\exists! x \in \{1,2,3\}$, $3x + 1$ is prime.

4. Fill in the following truth table for all possible values of P, Q and R.

P	Q	R	$P \wedge R$	$Q \to (P \wedge R)$	$(Q \to P) \wedge R$	$R \vee (P \to Q)$

5. Distribute the negation signs for each of the following, adjusting other symbols accordingly.

   (a) $\sim(\forall x, P(x) \wedge P(x+1))$

   (b) $\sim(\exists x, Q(x) \to Q(x+1))$

   (c) $\sim(\exists x, \forall y P(x,y) \vee Q(x,y))$

   (d) $\sim(\forall x, \exists y P(x,y) \wedge Q(x,y))$

   (e) $\sim(\forall x, \exists y P(x,y) \leftrightarrow Q(x,y))$

6. Assume a_n is a sequence of real numbers. The formal definition that a_n converges to $a_0 \in \mathbb{R}$ as $n \to \infty$ is:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, (n \geq N \to |a_n - a_0| < \epsilon)$$

   Negate this statement.

7. Negate the following, writing your results in english:

   (a) There was once a year in which every day was rainy or snowy.

   (b) For every week there is at least one day where if it's cloudy then it snows.

8. Provide proofs with justification of each of the following. Some statistics to help:
   - One is trivially true.
   - One is vacuously true.
   - Two should have direct proofs.
   - Two should have proofs of the contrapositive.
   - One requires an intermediate step by the contrapositive with a link to a direct proof.
   - One requires cases.

   (a) If $n, m \in \mathbb{Z}$ are both odd then $3n - m + 1$ is odd.
   (b) If $n \in \mathbb{Z}$ and $3n - 7$ is odd then $\frac{n}{2} + 1 \in \mathbb{Z}$.
   (c) If $x \in \mathbb{R}$ and $x^2 + 2x \leq 3$ then $-3 \leq x \leq 1$.
   (d) If $x \in \mathbb{R}$ and $|x + 1| + 1 = 0$ then $x^2 = 4$.
   (e) If $n \in \mathbb{Z}$ and $3n + 1$ is odd then n is even.
   (f) If $n \in \mathbb{Z}$ and $n^2 + n < 0$ then $|n + 1| + 1 > 0$.
   (g) If $n \in \mathbb{Z}$ then $n^2 + n + 1$ is odd.
   (h) If $f(x)$ is a function and $f'(x) - 2f(x) = 0$ then $f(x) \neq \sin(2x)$.

9. Explain why the following proofs fail. Explanations should be in full sentences with minimal notation.

   (a) Claim: If $x^2 - 4 = 0$ then $x = 2$.
   "Proof": Suppose $x = 2$. Then $x^2 = 4$ and so $x^2 - 4 = 0$.
   (b) Claim: $3 = -3$.
   "Proof": Let $x = 3$. Then $x^2 = (-x)^2$ so $\sqrt{x^2} = \sqrt{(-x)^2}$ and so canceling the square root and the square yields $x = -x$ and so $3 = -3$.
   (c) Claim: $1 = -1$.
   "Proof": $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1$.