Math 461 HW 5.

QR decomposition

If A is an $m \times n$ matrix, then matlab command $[P\ S] = qr(A)$ will return an $m \times m$ ortogonal matrix P and an $m \times n$ upper triangular matrix S so that $A = PS$.
Matlab gives a more general QR decomposition than the one described in Lay. In Lay A must have rank n. Matlab command does not require that.
If A has rank n then the first n columns of P form an orthonormal basis for $Col(A)$ and the last $m - n$ columns of P form an orthonormal basis for orthogonal complement of $Col(A)$, which is $Nul(A^T)$. The last $m - n$ rows of S will be zero.
Q in Lay coincides with the first n columns of P and R in Lay coincides with the first n rows of S.

Least squares solutions

If A is not a square matrix then $A\backslash b$ always gives a least squares solution to $Ax = b$.
However in the case when A is a square matrix and is not invertible matlab will not return a least squares solution. In the case when A is a square matrix you can get around this by adding a last column of zeroes to A. For example if A is 6×6 use $B = [A\ zeros(6, 1)]$. Then calculate $x = B\backslash b$ and delete the last entry of x. That will be a least squares solution to $Ax = b$.

Singular value decomposition.
For an $m \times n$ matrix A the command $[U\ S\ V] = svd(A)$ finds the singular value decomposition of A. That means $A = USV^T$ where S is a diagonal $m \times n$ matrix . If A is a real matrix, then U and V are orthogonal matrices. If A has rank r, then then S has an $r \times r$ diagonal matrix in its upper left corner and the rest of S is zero.

Other commands .
For any matrix A the command $orth(A)$ gives a matrix whose columns form an orthonormal basis for $Col(A)$.
$Null(A)$ gives a matrix whose columns form an orthonormal basis for $Nul(A)$.

Generate a random 5×5 matrix A with rank 3. Check that its rank indeed equals 3 before proceeding.

Problem 1. Generate a random vector b in $\mathbf{R}^5$. Find a least squares solution $\hat{x}$ to $Ax = b$. Compute the error vector $b - A\hat{x}$. Check that this vector is perpendicular to the column space of A. The error is the norm $||b - A\hat{x}||$. Check that the error is minimized by the least squares solution by computing the quantity $||b-Ax||$ for several random vectors x and comparing it to $||b-A\hat{x}||$.

Problem 2. Find a matlab QR decomposition of A and check that $A = QR$ and that Q is an orthogonal matrix .
Make a matrix B out of the first three columns of Q, and check whether they form an orthonormal set.
Next make a matrix N out of the last two columns of Q, and check whether they form an orthonormal set.
Also check that columns of N are perpendicular to columns of A.

Problem 3. Find the singular value decomposition of A.
Also find $orth(A)$ and $Null(A)$.
Deduce how matlab computes $orth(A)$ and $Null(A)$.