Simple Quadrature Rules

Single Panel Midpoint Rule

$$\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2}) + E_M(f)$$

where

$$E_M(f) = \frac{(b-a)^3}{24} f''(\xi)$$

for some ξ in the interval $[a,b]$.

Composite Midpoint Rule. Let $h = (b-a)/n$, and $x_j = a + j * h,\ j = 0, \ldots, n$ and let $s_j = (1/2)(x_{j-1} + x_j), j = 1, \ldots, n$ be the midpoint of each subinterval. Then

$$\int_a^b f(x)dx = h[f(s1) + f(s2) + \cdots + f(s_n)] + E_M(f,h)$$

where

$$E_M(f,h) = \frac{h^2(b-a)}{24} f''(\eta)$$

for some η in the interval $[a,b]$.

Single Panel Trapezoid Rule

$$\int_a^b f(x)dx = \frac{h}{2}(f(a) + f(b)) + E_T(f)$$

where

$$E_T(f) = -\frac{(b-a)^3}{12} f''(\xi)$$

for some ξ in the interval $[a,b]$.

Composite Trapezoid Rule

$$\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] + E_T(f,h)$$

where

$$E_T(f,h) = -\frac{h^2(b-a)}{12} f''(\eta)$$

for some η in the interval $[a,b]$.

Single Panel Simpson Rule

$$\int_a^b f(x)dx = \frac{h}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)] + E_S(f)$$

where

$$E_S(f) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

for some ξ in $[a, b]$.

Composite Simpson Rule Assume n is even. Then

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{n-1})+f(x_n)]+E_S(f,h)$$

where

$$E_S(f,h) = -\frac{(b-a)h^4}{180}f^{(4)}(\eta)$$

for some η in $[a, b]$.

Simpson's rule is related to the trapezoid rule and the midpoint rule by the equation

$$S(f) = \frac{2}{3}M(f) + \frac{1}{3}T(f).$$

The midpoint rule and the trapezoid rule are both exact on polynomials of degree ≤ 1, but not exact on x^2. Hence the midpoint rule and trapezoid rule and both of order 2. Simpson's rule is exact is exact on polynomials of degree ≤ 3, but not on x^4. Hence Simpson's rule is of order 4.

Estimates of the error. Use $T_n(f)$ to denote $T(f, h)$ where $h = (b-a)/n$. From the form of the error for T_n, assuming f'' changes slowly, we can deduce that

$$|E_T(f,n)| \approx \frac{4}{3}|T_n(f) - T_{2n}(f)|.$$

Note that we can write

$$T_{2n}(f) = \frac{1}{2}T_n((f) + \frac{h}{2}[f(s_1) + \cdots + f(s_n)$$

where s_j is the midpoint of the n^{th} subinterval.

For Simpson's rule we can deduce that

$$|E_S(f,n)| \approx \frac{16}{15}|S_n(f) - S_{2n}(f)|.$$

We can get a new rule of higher order by forming an average of $S(f, n)$ and $S(f, 2n)$. In fact,

$$Q(f) = \frac{16S(f,2n) - S(f,n)}{15}$$

has order 6. It is a five point Newton Cotes rule.

Gaussian Quadrature

We begin with a three point rule on $[-1, 1]$. Let

$$Q(g) = A_1g(t_1) + A_2g(t_2) + A_3g(t_3).$$

We want to choose the weights A_1, A_2, A_3 and the nodes t_1, t_2, t_3 so as to maximize the order of Q. To economize in the derivation, we shall assume some symmmetry: We take $t_1 = -t_3$ and $t_2 = 0$, and $A_1 = A_3$. Thus our rule becomes

$$Q(g) = A_1 g(-t_3) + A_2 g(0) + A_1 g(t_3).$$

With this symmetry, we have that for any power t^p with p odd, $Q(t^p) = 0 = \int_{-1}^{1} t^p dt$. Thus we shall determine A_1, A_2 and t_3 by requiring that Q be exact on the even powers $g(t) \equiv 1$,$g(t) = t^2$ and $g(t) = t^4$. This yields the equations

$$Q(1) = A_1 + A_2 + A_1 = \int_{-1}^{1} 1 dt = 2$$

or

$$2A_1 + A_2 = 2. \tag{1}$$

$$Q(t^2) = A_1 t_3^2 + A_1 t_3^2 = \int_{-1}^{1} t^2 dt = 2/3$$

or

$$A_1 t_3^2 = 1/3. \tag{2}$$

Finally we require that

$$Q(t^4) = A_1 t_3^4 + A_1 t_3^4 = \int_{-1}^{1} t^4 dt = 2/5$$

or

$$A_1 t_3^4 = 1/5. \tag{3}$$

Dividing equation (3) by equation (2), we find $t_3^2 = 3/5$, whence $t_3 = \sqrt{3/5}$. Substitution of this value of t_3 into (2) yields $A_1 = 5/9$, and finally (1) yields $A_2 = 8/9$. Hence our three-point Gaussian quadrature rule is

$$G_3(g) = \frac{5}{9} g(-\sqrt{3/5}) + \frac{8}{9} g(0) + \frac{5}{9} g(\sqrt{3/5}).$$

It has order 6 because it integrates exactly $g(t) = 1, t, t^2, t^3, t^4$ and t^5, but not $g(t) = t^6$. Use the map $\varphi(t) = a + \frac{(b-a)}{2}(t+1)$ to make the change of variable from $[-1, 1]$ to $[a, b]$. If f given on $[a, b]$, set $g(t) = f(\varphi(t))$. Then

$$G_3(f) = G_3(g) = \frac{(b-a)}{2}[\frac{5}{9} f(\varphi(-\sqrt{3/5}) + \frac{8}{9} f(\varphi(0)) + \frac{5}{9} f(\varphi(\sqrt{3/5})]$$

and the error is

$$E_3(f) = [\frac{(b-a)}{2}]^7 \frac{1}{15750} f^{(6)}(\eta).$$

There are Gaussian quadrature rules of all orders. If there are n weights and n nodes to be chosen, there are $2n$ degrees of freedom, and we can choose them so that G_n integrates all polynomials of degree $\leq 2n - 1$, but not of degree $2n$. Hence G_n is order $p = 2n$. The nodes of G_n are the zeros of the Legendre polynomial θ_n of degree n. The weights and nodes for Gaussian quadrature can easily be found on the web.

Lobatto quadrature

Lobatto quadrature using 4 points is a rule similiar to Gaussian quadrature but which uses the end points of the interval. On the interval $[-1, 1]$, the rule is

$$L_4(g) = A_1 g(-1) + A_2 g(-t_1) + A_2(t_1) + A_1 g(1).$$

Because of symmetry, L_4 is already exact on all odd powers t^p. We impose the conditions that L_4 be exact on the polynomials $1, t^2$ and t^4. This yields the equations

$$A_1 + A_2 = 1$$

$$A_1 + A_1 t_1^2 = \frac{1}{3}$$

and

$$A_1 + A_2 t_1^4 = \frac{1}{5}.$$

The solutions for A_1, A_2 and t_1 yield the Lobatto rule

$$L_4(g) = \frac{1}{6} g(-1) + \frac{5}{6} g(-\sqrt{1/5}) + \frac{5}{6} g(\sqrt{1/5}) + \frac{1}{6} g(1).$$

The Lobatto rule L_4 has order $p = 6$. There are Lobatto rules of all orders. The interior nodes $t_i, i = 1, \ldots, n-2$ are the zeros of the derivatives θ_n' of the Legendre polynomials θ_n.