

AMSC/CMSC 466 Problem set 5

1. The single panel midpoint rule is simply

$$M(f) = (b - a)f(m) \quad \text{where} \quad m = (a + b)/2.$$

a) Use a Taylor expansion of f about m to show that the error is

$$E = \int_a^b f dx - M(f) = \frac{(b - a)^3 f''(\xi)}{24}.$$

b) Derive the composite form of the midpoint rule along with the error. If we cut h in half, how does the error decrease?

c) Let $T(f) = (b - a)(f(a) + f(b))/2$ be the single panel trapezoid rule. Let f be strictly convex on $[a, b]$ (i.e., $f''(x) > 0$). Without using the forms of the error, explain why $M(f) < \int_a^b f dx < T(f)$.

d) What rule do we obtain if we take the weighted average $Q(f) = (1/3)T(f) + (2/3)M(f)$?

2. Write a MATLAB program to implement the trapezoid rule. Make it a function `trap(f,a,b,n)` where f is the function to be integrated over the interval $[a, b]$ and n is the number of subintervals. The function $f(x)$ should be written as inline function. The function mfile `trap.m` will use the instruction `feval`. This instruction passes the variable x to the named function with the command `feval(f,x)`. For more info, enter `help feval`. The call for `trap` will be `trap(f,a,b,n)`. Output should be the trapezoid estimate for the integral.

Apply the trapezoid rule to estimate each of the following integrals with $n = 2, 4, 8, 16, 32, 64$.

$$(i) \quad \frac{2}{\sqrt{\pi}} \int_0^1 e^{-x^2} dx = \text{erf}(1) \qquad (ii) \quad \int_0^1 x^{5/2} dx$$

$$(iii) \quad \int_{-4}^4 \frac{dx}{1 + x^2} \qquad (iv) \quad \int_0^{2\pi} \frac{dx}{2 + \cos(x)}.$$

(a) In cases (i), (ii), and (iii), compare your result with the exact answer (for case (i) this means compare with the MATLAB value for the error function). What power law governs the decrease of the error? Does this agree with the error behavior predicted by the error formula?

(b) In case (iv), we do not have an exact answer, but you will see that the results converge very quickly. Why?

3. Write a function mfile to implement Simpson's rule and apply it to the integrals of problem 2, same questions.

4. Now write a trapezoid program, `myquad`, with input a function $f(x)$ (given as an inline function), the interval $[a, b]$ and a tolerance ε . The call should be `myquad(f,a,b,tol)`. The program should apply the trapezoid rule, doubling the number n of intervals each time, until the estimated error

$$(4/3)|T_{2n}(f) - T_n(f)| < \varepsilon.$$

Use a while loop. The program should be very efficient, never having to compute a value of $f(x)$ more than once. Output should be the last value T_{2n} , the estimated error and the number n . Apply this function to each of the integrals (i), (ii), and (iii) of problem 1 with a tolerance of $\varepsilon = 10^{-6}$. Compare the estimated error with the actual error.

5. a) Find the coefficients A_0, A_1, A_2 such that the quadrature formula

$$Q(f) = A_0 f(-3/4) + A_1 f(0) + A_2 f(3/4)$$

is exact on the interval $[-1, 1]$ for all polynomials of degree ≤ 2 . What is the degree of this method?

b) Transform the method to the general interval $[a, b]$.

6. The weights and nodes for 4 point Gaussian quadrature on $[-1, 1]$ are given on page 498 of KC.

a) Transform to the interval $[a, b]$.

b) Write a function mfile `g4.m`, with call `g4(f,a,b)`, that implements Gaussian quadrature on the interval $[a, b]$ for the function $f(x)$, given as inline function.

c) Apply your Gaussian quadrature rule to the integral (i) of problem 2. Compare your results with trapezoid and Simpson's rule for $n = 4$.

7. Let $f(x) = \sqrt{x} \cos x$. Use the MATLAB routine `quad` to estimate

$$\int_0^1 f(x) dx.$$

For info on `quad`, enter `help quad`. You will see that there are various options. You can specify the tolerance and you can request the routine to display how many functions evaluations are made, and where. The default tolerance is 10^{-6} . If f is given as an inline function, the call is `quad(f, a, b, tol, trace)` where `trace` is any nonzero number. To use the default tolerance, the call is `quad(f, a, b, [], trace)`.

a) Use `quad` on this integral with the default tolerance. How many function evaluations are needed altogether? How many are needed to for the interval $[0, .00848675]$? How many are needed for the last interval $[.724842, 1]$?

b) Now make the change of variable $x = t^2$ in the integral and repeat your calculation with `quad`. What is the difference? You have “softened the singularity ” at $x = 0$ with the change of variable.