NOTES ON ACCELERATION AND CURVATURE

1. Acceleration

Let $\mathbf{r}(t)$ parametrize a smooth curve C. The velocity is $\mathbf{v}(t) = \mathbf{r}'(t)$ and the acceleration is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$. When we normalize the velocity, we obtain the *unit tangent vector* in the same direction:

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}.$$

We have

$$1 = \mathbf{T}(t) \cdot \mathbf{T}(t) \tag{1}$$

for all t. If we differentiate both sides of this equation we find that

$$\begin{aligned} 0 &= \frac{d}{dt}[\mathbf{T}(t) \cdot \mathbf{T}(t)] \\ &= \mathbf{T}'(t) \cdot \mathbf{T}(t) + \mathbf{T}(t) \cdot \mathbf{T}'(t) \\ &= 2\mathbf{T}'(t) \cdot \mathbf{T}(t). \end{aligned}$$

Thus $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are always orthogonal. We define the *unit normal vector*

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|.} \tag{2}$$

For each t, the point $\mathbf{r}(t)$ and the two orthogonal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$ span a plane $\mathcal{P}(t)$ called the *osculating* (" kissing ") *plane*. The osculating plane $\mathcal{P}(t)$ contains the unit tangent vector $\mathbf{T}(t)$. Hence for each t, the curve C is tangent to the plane $\mathcal{P}(t)$ at $\mathbf{r}(t)$. When the motion is two-dimensional, i.e., $z(t) \equiv 0$, $\mathcal{P}(t)$ is just the xy plane. Note, however, that the normal vector $\mathbf{N}(t)$ and the osculating plane $\mathcal{P}(t)$ are not defined when $\mathbf{T}'(t) = 0$.

It is not obvious, but very important, that the acceleration vector $\mathbf{a}(t)$ also lies in the osculating plane $\mathcal{P}(t)$. To see this, we shall show that $\mathbf{a}(t)$ can be written

$$\mathbf{a}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t) \tag{3}$$

for a unique choice of coefficients $a_T(t)$ and $a_N(t)$. Since $\mathbf{T}(t)$ and $\mathbf{N}(t)$ both lie in the osculating plane $\mathcal{P}(t)$, the same is true of $\mathbf{a}(t)$. To derive (3), we note that

$$\mathbf{v}(t) = \frac{\mathbf{v}(t)\ \|\mathbf{v}(t)\|}{\|\mathbf{v}(t)\|} = \|\mathbf{v}(t)\|\ \mathbf{T}(t). \tag{4}$$

Now we differentiate (4) to obtain

$$\begin{aligned}\mathbf{a}(t) &= \frac{d}{dt}\mathbf{v}(t) = (\frac{d}{dt}\|\mathbf{v}(t)\|)\ \mathbf{T}(t) + \|\mathbf{v}(t)\|\ \mathbf{T}'(t) \\ &= (\frac{d}{dt}\|\mathbf{v}(t)\|)\ \mathbf{T}(t) + \|\mathbf{v}(t)\|\ \|\mathbf{T}'(t)\|\ \mathbf{N}(t).\end{aligned}$$

Thus the coefficients in (3) are

$$a_T(t) = \frac{d}{dt}\|\mathbf{v}(t)\| \tag{5}$$

and

$$a_N(t) = \|\mathbf{v}(t)\|\ \|\mathbf{T}'(t)\|. \tag{6}$$

Note that $a_N \geq 0$. The magnitude of the tangential component of acceleration is $|a_T|$ while a_N is the magnitude of the normal component of acceleration. a_T is the rate of change of the speed while a_N expresses the rate of change of the direction of the velocity.

Because $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, we have

$$\|\mathbf{a}\|^2 = a_T^2 + a_N^2. \tag{7}$$

It can be rather tedious to compute a_T and a_N from (5) and (6) although it is usually easy to compute $\mathbf{v}$ and $\mathbf{a}$. Here are some alternate expressions for a_T and a_N. First we see that

$$\frac{d}{dt}\|\mathbf{v}(t)\|^2 = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v})(t) = 2\mathbf{v}(t)\cdot\mathbf{a}(t). \tag{8}$$

But using the chain rule we also have

$$\frac{d}{dt}\|\mathbf{v}(t)\|^2 = 2\|\mathbf{v}(t)\|\ \frac{d}{dt}\|\mathbf{v}(t)\|. \tag{9}$$

Equating (8) and (9) and dividing by $\|\mathbf{v}(t)\|$, we arrive at

$$a_T = \frac{d}{dt}\|\mathbf{v}(t)\| = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}(t)\|}. \tag{10}$$

Then use (7) to find a_N:

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}. \tag{11}$$

Another expression for a_N is derived from (3) by taking the cross product with $\mathbf{v}$. The cross product $\mathbf{v}\times\mathbf{T} = 0$ because $\mathbf{v}$ and $\mathbf{T}$ are parallel. Hence the cross product of (3) with $\mathbf{v}$ yields

$$\mathbf{a}\times\mathbf{v} = a_N\ \mathbf{N}\times\mathbf{v}.$$

Now $\mathbf{v}$ and $\mathbf{N}$ are orthogonal so that

$$\|\mathbf{a} \times \mathbf{v}\| = a_N \quad \|\mathbf{N} \times \mathbf{v}\| = a_N \|\mathbf{N}\| \; \|\mathbf{v}\| = a_N \|\mathbf{v}\|.$$

Finally we arrive at

$$a_N = \frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|}. \tag{12}$$

Example 1.1 Two-dimensional motion along a parabola. Consider the curve C in the xy plane given by the function $y = x^2/2$. The simplest, smooth, parameterization of C is

$$t \to \mathbf{r}(t) = (t, t^2/2).$$

Then $\mathbf{v}(t) = (1, t)$ and

$$\mathbf{T}(t) = \frac{(1,t)}{\sqrt{1+t^2}}$$

and

$$\mathbf{T}'(t) = \frac{(-t,1)}{(1+t^2)^{3/2}}.$$

Next we see that

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{(-t,1)}{\sqrt{1+t^2}}.$$

Now $\mathbf{a}(t) \equiv (0, 1)$ so that $\mathbf{a} \cdot \mathbf{v} = t$. It follows that

$$a_T = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}(t)\|} = \frac{t}{\sqrt{1+t^2}}.$$

Then using (7),

$$a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \frac{1}{\sqrt{1+t^2}}.$$

Hence, the decomposition (3) of the acceleration is

$$\mathbf{a} = (0,1) = (\frac{t}{\sqrt{1+t^2}})\frac{(1,t)}{\sqrt{1+t^2}} + (\frac{1}{\sqrt{1+t^2}})\frac{(-t,1)}{\sqrt{1+t^2}}.$$

Another smooth parametrization of C is

$$t \to (t + t^3, (t+t^3)^2/2).$$

If you calculate $\mathbf{v}$ and $\mathbf{a}$ in this parameterization, the results are different. However $\mathbf{T}$ and $\mathbf{N}$ turn out to be the same. We shall see later that $\mathbf{T}$ and $\mathbf{N}$ are independent of the parameterization. They are determined by the geometry of the curve.

Example 1.2 Circular motion in the plane. Let the C be the circle of radius ρ, centered at $(0,0)$. We shall parametrize C with

$$t \to \mathbf{r}(t) = \rho(\cos(\omega t),\ \sin(\omega t)), \qquad 0 \le t \le 2\pi/\omega.$$

This is actually a family of parameterizations depending on the *angular velocity* ω which has the units of radians/time. The velocity $\mathbf{v}(t) = \rho\omega(-\sin(\omega t),\ \cos(\omega t))$ and the speed $\|\mathbf{v}(t)\| = \rho\omega \equiv v_0$ is constant. The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = (-\sin(\omega t),\ \cos(\omega t))$$

and the unit normal vector is

$$\mathbf{N}(t) = (-\cos(\omega t),\ -\sin(\omega t)). \tag{13}$$

Since the speed is constant, $a_T = (d/dt)\|\mathbf{v}(t)\| = 0$. There is no accleration in the tangential direction. Hence the acceleration must be all in the normal direction. We calculate

$$\begin{aligned} \mathbf{a} &= \rho\omega^2(-\cos(\omega t),\ -\sin(\omega t)) \\ &= \rho\omega^2\mathbf{N} = \frac{v_0^2}{\rho}\mathbf{N}. \end{aligned}$$

Hence

$$a_N = \frac{v_0^2}{\rho}. \tag{14}$$

Example 1.3 The circular helix with a linear rise function. We take

$$\mathbf{r}(t) = \rho(\cos(\omega t),\ \sin(\omega t),\ ct).$$

The " rise function " is $z(t) = ct$ where $c > 0$. The velocity

$$\mathbf{v}(t) = (-\rho\omega\sin(\omega t),\ \rho\omega\sin(\omega t), c)$$

and the speed is again constant, $\|\mathbf{v}(t)\| = \sqrt{\rho^2\omega^2 + c^2}$. The acceleration is

$$\mathbf{a}(t) = \rho\omega^2(-\cos(\omega t),\ -\sin(\omega t), 0).$$

Again $a_T = (d/dt)\|\mathbf{v}(t)\| = 0$. Hence $\mathbf{a}(t) = a_N\mathbf{N}(t)$ with $a_N = \rho\omega^2$ and

$$\mathbf{N}(t) = (-\cos(\omega t),\ -\sin(\omega t),\ 0).$$

2. Curvature

Suppose a curve C is parameterized by $t \to \mathbf{r}(t)$. At each time t, a_N is a measure of the circular component of the motion. To see this idea more explicitly, we fix a time t_0 and let $P_0 = \mathbf{r}(t_0)$. We imagine a circle of radius ρ lying in the osculating plane $\mathcal{P}(t)$ when $t = t_0$. We take its center to be the point $P_0 + \rho\mathbf{N}(t_0)$. For any choice of ρ, this circle will be tangent to the curve C at P_0 (see Figure 1). If a point moves around the circle with angular velocity ω, its speed is $v_0 = \rho\omega$. We assume that the circle is parameterized so that the tangent vector to the circle at P_0 points in the same direction as $\mathbf{r}'(t_0)$. We shall choose the parameters v_0 and ρ so that the motion around the circle coincides with the circular component of the motion on the curve C at P_0. In physical terms, if we are riding a bicycle, and at time t_0 we fix the angle of the handle bars, and continue at a constant speed, what will be the circle followed by the bicycle ?

First we choose $v_0 = \|\mathbf{r}'(t_0)\|$. This will ensure that the motion around the circle and on C have the same velocity vector at P_0. Next we choose ρ so that the normal acceleration of the circular motion, given by (14), agrees with the normal acceleration of $\mathbf{r}(t)$ when $t = t_0$. Thus we set

$$\frac{v_0^2}{\rho} = \|\mathbf{r}'(t_0)\| \; \|\mathbf{T}'(t_0)\|.$$

Since $v_0 = \|\mathbf{r}'(t_0)\|$, the radius of the circle is found to be

$$\rho = \frac{\|\mathbf{r}'(t_0)\|}{\|\mathbf{T}'(t_0)\|}. \tag{15}$$

When v_0 and ρ are chosen this way, the circle is known as the *osculating circle.* The reciprocal of (15) is called the *curvature* at the point P_0 :

$$\kappa = \frac{\|\mathbf{T}'(t_0)\|}{\|\mathbf{r}'(t_0)\|}. \tag{16}$$

The radius ρ of the circle is called the *radius of curvature* at P_0.

Some alternate expressions for the curvature, that are easier to use for computations, come from (6) and (12). They are

$$\kappa = \frac{a_N}{\|\mathbf{v}\|^2} = \frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|^3}. \tag{17}$$

The second formula is probably the easiest to use. In two dimensions this formula reduces to

$$\frac{|x''y' - x'y''|}{[(x')^2 + (y')^2]^{3/2}}. \tag{18}$$

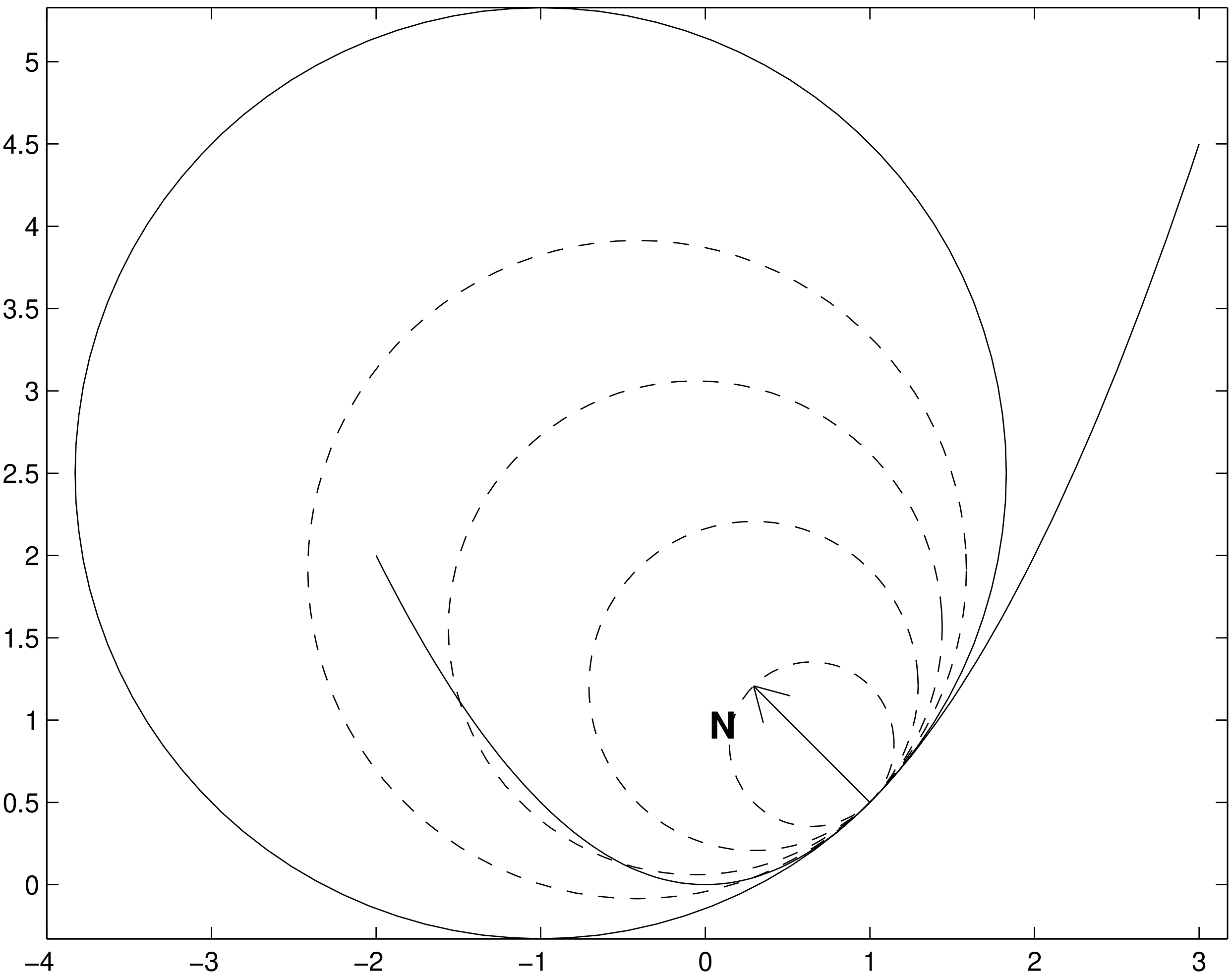

Figure 1: Curve $t \to (t, t^2/2)$ with several tangent circles at the point $P_0 = (1, 1/2)$. The osculating circle is the one in solid line.

In particular, we see that if C is the circle parameterized by

$$t \to \rho(\cos(\omega t),\ \sin(\omega t)), \quad 0 \leq t \leq 2\pi/\omega$$

then $\kappa = 1/\rho$ is independent of the angular velocity ω. This is an example of the fact that curvature is determined by the geometry of the curve, being independent of the parameterization.

To see that $\mathbf{T}, \mathbf{N}$, and κ are independent of the parameterization, we describe them in terms of parameterization with respect to arc length. Let $t \to \mathbf{r}(t)$ be a parameteriztion of a curve C, $a \leq t \leq b$. We define the arc length parameter by

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\ du. \tag{19}$$

We have $s(a) = 0$ and $s(b) = L$ where L is the length of the curve and $ds/dt = \|\mathbf{r}'(t)\|$. Let $s \to \mathbf{q}(s)$, $0 \leq s \leq L$ parameterize C with respect to arc length. Then

$$\mathbf{r}(t) = \mathbf{q}(s(t)). \tag{20}$$

For example, in the case of a circle of radius ρ, the parameterization with respect to arc length is

$$s \to \mathbf{q}(s) = \rho(\cos(s/\rho),\ \sin(s/\rho)).$$

We differentiate (20) to find

$$\mathbf{r}'(t) = \mathbf{q}'(s(t))\ ds/dt = \mathbf{q}'(s(t))\ \|\mathbf{r}'(t)\|. \tag{21}$$

Hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{q}'(s(t)). \tag{22}$$

Thus $\mathbf{q}'(s)$ is always a unit tangent vector. To see how $\mathbf{N}$ is expressed in terms of arc length, differentiate (22):

$$\mathbf{T}'(t) = \mathbf{q}''(s(t))\ \|\mathbf{r}'(t)\|. \tag{23}$$

We have

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\mathbf{q}''(s(t))}{\|\mathbf{q}''(s(t))\|}.$$

Finally we see from (21) and (23) that κ has a very simple expression in terms of arc length:

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \|\mathbf{q}''(s(t))\|\ . \tag{24}$$

Since $\mathbf{r}(t)$ was an arbitrary parameterization of C, we conclude that κ is independent of the parameterization.

As a consequence, we can characterize the osculating circle at the point P_0 on C as follows. Let $s \to \mathbf{q}(s)$ be the parameterization of C with respect to arc length, and let $s \to \mathbf{p}(s)$ be the parameterization of the osculating circle with respect to arc length. Then

$$\begin{aligned} \mathbf{p}(s_0) &= \mathbf{q}(s_0) \\ \mathbf{p}'(s_0) &= \mathbf{q}'(s_0). \end{aligned}$$

These two equations express the fact the osculating circle at P_0 is tangent to the curve C at P_0. But in addition, the radius ρ is chosen so that

$$\mathbf{p}''(s_0) = \mathbf{q}''(s_0).$$

This means that near P_0 the osculating circle is a very good approximation to the curve C.

When the curve is parameterized with respect to arc length, the formula (24) for κ is very simple, $\kappa = \|\mathbf{q}''(s)\|$. However, the parameterization with respect to arc length may be very difficult or impossible to compute. It may be more natural to use another parameterization.

Example 2.1 We return to example 1.1 where the curve C is the graph of the parabola $y = x^2/2$. We shall use the parameterization

$$t \to (t, t^2/2)$$

whence $\mathbf{r}'(t) = \mathbf{v}(t) = (1, t)$ and $\mathbf{a} = (0, 1)$. We use (18) to compute κ. We have

$$x''y' - s'y'' = -1$$

so that

$$\kappa = \frac{1}{(1 + t^2)^{3/2}}.$$

Obviously κ has its largest value, $\kappa = 1$ when $t = 0$, and κ tends to zero as $t \to 0$.

Example 2.2 Consider the ellipse whose equation is

$$x^2 + y^2/4 = 1.$$

A convenient parameterization is

$$\mathbf{r}(t) = (\cos t,\ 2\sin t), \quad 0 \leq t \leq 2\pi.$$

Since $\mathbf{v}(t) = (-\sin t,\ 2\cos t)$ and $\mathbf{a} = (-\cos t,\ -2\sin t)$, we calculate κ from (18).

$$\begin{aligned} x''y' - x'y'' &= (-\cos t)(2\cos t) - (-\sin t)(-2\sin t) \\ &= -2(\cos^2 t + \sin^2 t) = -2. \end{aligned}$$

Hence

$$\kappa = \frac{2}{[\sin^2 t + 4\cos^2 t]^{3/2}}.$$

κ is largest, $\kappa = 2$, when $t = \pi/2$ or $t = 3\pi/2$ (at the small ends of the ellipse). It is smallest, $\kappa = 1/4$, when $t = 0, \pi$.

Example 2.3 The circular helix, parameterized by $\mathbf{r}(t) = (\cos t,\ \sin t,\ ct)$ where $c > 0$. As we saw before, $\mathbf{v}(t) = (-\sin t,\ \cos t,\ c)$ and $\mathbf{a} = (-\cos t,\ -\sin t,\ 0)$. Consequently

$$\mathbf{a} \times \mathbf{v} = (c\sin t,\ -c\cos t,\ 1)$$

so that $\|\mathbf{a} \times \mathbf{v}\| = \sqrt{1 + c^2}$. Finally, using (17),

$$\kappa = \frac{\|\mathbf{a} \times \mathbf{v}\|}{\|\mathbf{v}\|^3} = \frac{\sqrt{1+c^2}}{(1+c^2)^{3/2}} = \frac{1}{1+c^2}.$$

Example 2.4 Consider the curve parameterized by

$$t \to (t^2 \cos t,\ t^2 \sin t, 2 - t).$$

We see that

$$\mathbf{v} = (2t\cos t - t^2 \sin t,\ 2t\sin t + t^2\cos t,\ -1)$$

and

$$\mathbf{a} = (2\cos t - 4t\sin t - t^2\cos t,\ 2\sin t + 4t\cos t,\ 0).$$

We shall calculate $\mathbf{T}$, $\mathbf{N}$ and the curvature. Now

$$\|\mathbf{v}\|^2 = (2t\cos t - t^2\sin t)^2 + (2t\sin t + t^2\cos t)^2 + 1$$

and using the identity $\sin^2 t + \cos^2 t = 1$ several times we find that

$$\|\mathbf{v}(t)\| = \sqrt{1 + 4t^2 + t^4}.$$

Hence

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(2t\cos t - t^2\sin t, 2t\sin t + t^2\cos t, -1)}{\sqrt{1+4t^2+t^4}}. \tag{25}$$

Calculating $\mathbf{T}'$ from this expression would be a bear. Instead we note that $\mathbf{T} = \mathbf{v}/||\mathbf{v}||$ so that

$$\begin{aligned}
\mathbf{T}' &= \frac{\mathbf{a}}{||\mathbf{v}||} + \mathbf{v}\frac{d}{dt}(\frac{1}{||\mathbf{v}||}) \\
&= \frac{\mathbf{a}}{||\mathbf{v}||} - \frac{\mathbf{v}}{||\mathbf{v}||^2}\frac{d}{dt}||\mathbf{v}|| \\
&= \frac{\mathbf{a}}{||\mathbf{v}||} - \frac{\mathbf{v}}{||\mathbf{v}||^2}\frac{\mathbf{a}\cdot\mathbf{v}}{||\mathbf{v}||} \\
&= \frac{\mathbf{a}||\mathbf{v}||^2 - (\mathbf{a}\cdot\mathbf{v})\mathbf{v}}{||\mathbf{v}||^3}.
\end{aligned}$$

Therefore

$$\mathbf{N} = \frac{\mathbf{T}'}{||\mathbf{T}'||} = \frac{\mathbf{a}||\mathbf{v}||^2 - (\mathbf{a}\cdot\mathbf{v})\mathbf{v}}{||\mathbf{a}||\mathbf{v}||^2 - (\mathbf{a}\cdot\mathbf{v})\mathbf{v}||} \tag{26}$$

Now at least we can find $\mathbf{T}$ and $\mathbf{N}$ at $t = 0$. We have

$$\mathbf{T}(0) = \mathbf{v}(0) = (0, 0, -1)$$

and since $\mathbf{a}(0) = (2, 0, 0)$, formula (26) implies that

$$\mathbf{N}(0) = \frac{\mathbf{a}(0)}{||\mathbf{a}(0)||} = (1, 0, 0).$$

Hence the osculating plane at $(0, 0, 0)$ is the xz plane and the osculating circle has its center on the positive x axis. We find the radius of the osculating circle by calculating the curvature. The vector product $\mathbf{a} \times \mathbf{v}$ is not too hard to calculate, and we find that

$$||\mathbf{a} \times \mathbf{v}|| = \sqrt{4 + 12t^2 + 37t^4 + 12t^6 + t^8}.$$

The curvature is therefore

$$\kappa = \frac{||\mathbf{a} \times \mathbf{v}||}{||\mathbf{v}||^3} = \frac{\sqrt{4 + 12t^2 + 27t^4 + 12t^6 + t^8}}{(1 + 4t^2 + t^4)^{3/2}}.$$

In particular, $\kappa(0) = 2$ so that the osculating circle at $(0, 0, 0)$ has radius $1/2$ and is centered at the point $(1/2, 0, 0)$.