MATH 401 Applications of linear algebra
Problem set 2
Networks and Structures

1. Consider the network of four towns and six roads here.

a) Write an incident matrix A to describe this network.

Suppose that some commodity, like TV sets, is produced or consumed at each town. Let f_j be the surplus or deficit of TV sets at town j. Remember that $f_j > 0$ means a deficit, and $f_j < 0$ means a surplus .

Let x_j be the price of TV sets at town j, $j = 1, 2, 3, 4$. Let y_i be rate of flow of TV sets along road i and let v_i be the price difference along road i, $i = 1, 2, 3, 4, 5, 6$.

b) Write down the matrix equations relating x and v. What conditions must be satisfied by v for there to exist a solution x?

c) Write down the matrix equations relating y and f. What condition must $f = (f_1, f_2, f_3, f_4)$ satisfy for there to exist a solution y?

d) Now suppose that the shipping conditions permit a certain number y_i of TV sets to be shipped each week on road i. Let $c_i > 0$ be the shipping capacity of each road. Explain why in this context, the relation is $y_i = c_i v_i$. (Recall that for the water pipes network, $y_i = -c_i v_i$.)

Write the three sets of equations which govern a steady circulation of TV sets in this network. Then reduce to a single equation $Bx = f$. How can you modify this system so that for a given set of f_j, there is a unique set of prices x_j?

e) Let $c_1 = .5,\ c_2 = 1,\ c_3 = 1.5,\ c_4 = 2,\ c_5 = .4,\ c_6 = 1.7$.

What is the matrix B in this case. You may use matlab to multiply the matrices. What is the reduced 3×3 matrix $\tilde{B}$? Let $f_1 = -100, f_2 = 50, f_3 = -200$. What is f_4? Let $\tilde{x} = (x_1, x_2, x_3)$ and $\tilde{f} = (f_1, f_2, f_3)$. Solve the system $\tilde{B}\tilde{x} = \tilde{f}$.

f) What are the prices x_j if they are normalized so that $x_4 = 100$? In which town is the price the least? In which town is the price the highest? Along which road is the flow the greatest? Do these results agree with your understanding of supply and demand?

2. Consider the pin jointed framework shown here. Joints A and B are fixed joints. The others are free joints.

a) Forces $f_{left} = (f_1, f_2)$ and $f_{right} = (f_3, f_4)$ are applied at the two vertices as shown. What is the balance of force law at each free joint? What is the matrix A such that $A^T y = f$?

b) What is the matrix equation relating the infinitesimal displacements $x = (x_1, x_2)$ at the left free joint, and $x = (x_3, x_4)$ at the right free joint, with the elongations (strains) e_i in each beam ?

c) Assume Hooke's law relates the elongations (strains) e_i and the stresses y_i in each beam, with $e_i = k_i y_i$. Write down the three sets of equations that describe the equilibrium of this system when forces f_{left} and f_{right} are applied to the free joints.

d) Reduce the three sets of equations to a single matrix equation for x in terms of f. Will there be a unique solution x for each f?

e) Let $k_1 = 2,\ k_2 = 1.5,\ k_3 = 1,\ k_4 = 1.2$. Compute the matrix $B = A^T K^{-1} A$ and its inverse B^{-1}. Let $f = (-1, 1, 1, -1)$. What is the vector x of displacements? What is the vector of elongations e? What is the vector of stresses y?