

MATH 401 Applications of linear algebra  
Problem set 3  
Problems on Least Squares and on  
Discrete Fourier Transform

**1.** The first three Legendre polynomials on  $[-1, 1]$  are  $u_0(x) = 1$ ,  $u_1(x) = x$  and  $u_2(x) = x^2 - 1/3$ .

a) Find the polynomial  $q_2(x)$  of degree  $\leq 2$  which is the best approximation to  $f(x) = e^x$  in the least squares sense on the interval  $[-1, 1]$ .

b) Plot  $e^x$  and  $q(x)$  together on  $[-1, 1]$ .

**2.** In this exercise we shall find the coefficients of the line that makes the best fit in the least squares sense to some randomly generated data. First, we do it the long way, solving the normal equations. Then we do it using the specialized MATLAB commands.

The data will be a random perturbation of points on the graph of  $f(x) = 1/x$ . We use the MATLAB command `rand(m,n)` which generates an  $m \times n$  matrix of random numbers in the interval  $[0, 1]$ . Here is a set of instructions to generate the data points. The first command generates the  $x$  values of the data points in a column vector. The second command produces the random  $y$  values.

```
>> x = linspace(1, 10, 21)';  
>> y = x.^(-1) + .25*rand(21,1);
```

We construct the matrix  $A$  with the commands

```
>> col = ones(21,1);  
>> A = [col, x];
```

Now solve the normal equations  $A^T A c = A^T y$  and record the coefficients  $c = (c_0, c_1)$ .

Next we find the same coefficients using the MATLAB commands `polyfit` and `polyval`.

```
% calculate the coefficients for the line least squares fit.  
>> p1 = polyfit(x,y,1)
```

The entries in `p1` should be the same as the coefficients  $c$ , but in opposite order. Finally we plot the data and the line.

```
% These are the plotting points.
>> xval = linspace( 1, 10, 201);
% calculate the values of the line at the plotting points.
>> yval1 = polyval(p1, xval);
>> plot(x,y, '*')
>> hold on
>> plot(xval, yval1)
```

In the same figure, we shall plot the parabola that fits the data best in the least squares sense.

```
% calculate the coefficients for a parabola least squares fit.
>> p2 = polyfit(x,y,2);
>> yval2 = polyval(p2,xval);
>> plot(xval, yval2)
```

Hand in the plot with data points, the line, and the parabola. Which of the two, the line or the parabola, gives the best fit on the interval  $[1, 10]$ ? Which of the two makes the better prediction of the value at  $x = 11$ ?

3. Let  $A$  be the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 3 & 0 \end{bmatrix}.$$

Let  $b = (1, 1, 1, 1, 1)$ .

a) Enter the matrix  $A$  in MATLAB and the vector  $b$  as a column vector. Use the operation `x0 = A\b`. Does  $Ax_0 = b$ ? Calculate  $\|Ax_0 - b\|$  with the command `norm(A*x0 - b)`.

b) Now write a loop that generates random column vectors  $x_j \in R^3$  and computes  $\|Ax_j - b\|$  for  $j = 1, \dots, 10$ . Compare these results with  $\|Ax_0 - b\|$ . What kind of solution does the MATLAB command `A\b` produce in this situation?

4. Problem on DFT Let

$$f(t) = 2 \exp(-2it) - 1 + \exp(3it).$$

a) Let  $N = 4$ . Write out the equations for the coefficients  $d_k$ ,  $k = 0, 1, 2, 3$ . Solve for  $d_k$  by inverting the Fourier matrix  $F_4$ .

b) Write out the real and imaginary parts of the DFT approximation

$$g(t) = d_0 + d_3e^{-it} + d_1e^{it} + d_2e^{2it}.$$

Use MATLAB to plot the real parts of  $f$  and  $g$  together on  $[0, 2\pi]$  in one graph, and the imaginary parts of  $f$  and  $g$  together in another. Circle the points where  $g$  interpolates  $f$ .

c) Now let  $N = 6$ . Find the primitive complex root of unity  $W_6$ . Use MATLAB to calculate the Fourier matrix  $F_6$ .

d) Make an inline function for  $f$ . Solve the system  $F_6d = f$  using the following instructions.

```

    sample = linspace(0, 2*pi, 7);
% We do not use the last sample point which is 2*pi
% transpose makes it into a column vector
    tt = transpose(sample(1:6));
% ff is also a column vector
    ff = f(tt);
% Enter here the Fourier matrix F = F_6.
    dd = F\ff;
    d(0) = dd(1);
    d = dd(2:N);
% sum up the 6 terms in g using the coefficients d(k).

```

Compare  $g(t)$  and  $f(t)$ . They should agree.

5. Let  $f(t) = \exp(\sin(2t)) + \cos(10t)$ . Go to my webpage at [www.math.umd.edu/~jec](http://www.math.umd.edu/~jec) and go to Math 401, and then to demos where you will find the Mfile `dft.m`.

a) Use this program first with  $N = 6$ . Print out the graphs. Circle the points where  $g$  interpolates  $f$ .

b) Keep increasing  $N$  until you get a good fit for both real and imaginary parts. How large does  $N$  have to be? How does the number of sampling points relate to the number of oscillations of  $f$  in the interval  $[0, 2\pi]$ ?