MATH 401 Applications of linear algebra
Problem set 4
Problems on eigenvalues and Markov processes

**1.** In this exercise we use the MATLAB codes `gersh1` and `gersh2` which are available on my web page.

a) Consider the matric

$$A = \begin{bmatrix} 1 & s \\ -s & 2 \end{bmatrix}.$$

$A$ has the form $A = L + sR$ where $L$ is diagonal and $R$ is skew symmetric. For what values of $s$ are the eigenvalues of $A$ real? What is the value $s^*$ where the real eigenvalues coalesce and then split into two complex eigenvalues? Plot the eignevalues in the complex plane for $0 \leq s \leq 1$.

The codes `gersh1` and `gersh2` start with the diagonal matrix $L$, with elements $1, 2, 4, 7$, and adds a perturbation $sR$ where $R$ is a random matrix. The eigenvalues of $A(s) = L + sR$ are plotted in red for $0 \leq s \leq 1$. Eigenvalues of $A(0) = L$ are located in blue and the Gershgorin disks for the matrix $A(1) = L + R$ are plotted as blue circles.

b) In the code `gersh1`, the perturbation $R$ is a random matrix with all entries between 0 and 1. Run this code several times to see how the eigenvalues change. Do the eigenvalues of $A(1) = L + R$ ever come close to the boundaries of the Gershgorin disks?

c) In the code `gersh2`, the perturbation matrix $R$ is a random skew symmetric matrix with entries between $-1$ and 1. Run this code several times. Do you see behavior similar to that of part a)? Do the eigenvalues of $A(1)$ come close to the boundaries of the Gershgorin disks?

**2.** Use the following MATAB commands to generate a $10 \times 10$ Markov matrix.

```
A = rand(10,10);
for j = 1:10
  colsum = sum(A(:,j));
  A(:,j) = A(:,j)/colsum;
end
```

a) Use the command `[S,D] = eig(A)` to find the steady state of this process. Normalize the steady state so that the entries sum to one:
`steady = steady/sum(steady)`

b) Set `d = diag(D)` and then look at `abs(d)` to find the eigenvalue $\lambda_2$ with largest absolute value, $|\lambda_2| < 1$.

c) Now generate a random column vector having entries that sum to one.

```
v = rand(10,1);
v = v/sum(v);
```

We know that for any normalized vector $v$, the iterates $A^k v$ converge to the steady state as $k \to \infty$. Can we estimate how fast? Combine the all the previous instructions into a code that compares the value $|\lambda_2|^k$ with the difference `steady - (A^k)*v` for several different matrices $A$, several different vectors $v$, and the values $k = 5, 10, 20$. What do you observe?

**3.** Write a MATLAB code that generates a random $4 \times 4$ matrix $A$ and a random $4 \times 1$ column starting vector $u_0$. Then write a loop to implement the power method to approximate the largest eigenvalue of $A$. Try 10 iterations. Compare your result with the result you get by using MATLAB command `eig`.